OTIS Practice Exam Solutions

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EXAM-SAMPLE-02-SOLN

> Solve $f(m+n) = f(m) + f(n) + mn$ for $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$.

Let $c = f(1)$. Setting $m = 1$ we get $f(n+1) = f(n) + n + c$. Hence,

$$f(n) = (1 + \cdots + (n-1)) + nc = \binom{n}{2} + cn$$

for all n, by induction. This indeed works.

> Let $ABCD$ be a convex quadrilateral. Assume that the incircle of triangle ABD is tangent to $\overline{AB}$, $\overline{AD}$, $\overline{BD}$ at points W, Z, K. Also assume that the incircle of triangle CBD is tangent to $\overline{CB}$, $\overline{CD}$, $\overline{BD}$ at points X, Y, K. Prove that quadrilateral $WXYZ$ is cyclic.

From the concurrence of the Gregonne point, it follows that lines WZ, XY, and BD concur at the harmonic conjugate T of K with respect to $\overline{BC}$. (One can also see the concurrence directly by applying Ceva and Menelaus.)

Then $TK^2 = TW \cdot TZ = TX \cdot TY$, as desired.

Alternatively, inversion at T sends $WXYZ$ to a rectange, since the two incircles become parallel lines, while W^*X^* is the perpendicular bisector of B^*K, etc.

> Positive integers $x_1, x_2, \ldots, x_n$ ($n \ge 4$) are arranged in a circle such that each x_i divides the sum of the neighbors; that is,
>
> $$\frac{x_{i-1} + x_{i+1}}{x_i} = k_i$$
>
> is an integer for each i, where $x_0 = x_n$, $x_{n+1} = x_1$. Prove that
>
> $$2 \le \frac{k_1 + \cdots + k_n}{n} < 3.$$

Lower bound is AM-GM.

For the upper bound, we prove $k_1 + \cdots + k_n \le 3n - 1$ by induction on $n \ge 3$. The base case $n = 3$ is left as an exercise. There are two cases:

- If all x_i are equal, then $k_1 + \cdots + k_n = 2n \le 3n - 1$.
- Otherwise, let i be such that x_i is maximal. This requires $x_{i-1} + x_{i+1} = x_i$. Then deleting x_i gives a working configuration of $n - 1$ numbers around a circle, and decreases $\sum k_i$ by 3, completing the induction.

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Remark. Examples of equality cases, which motivate this solution:

- $(1, 2, 3, 4, 5, 6, 7)$.
- $(1, 2, 3, 4, 5, 6, 13, 7)$ (the one I found during exam).
- $(1, 4, 7, 3, 8, 13, 5, 7, 2)$.

Let m and s be positive integers with $2 \le s \le 3m^2$. Define a sequence $a_1, a_2, \dots$ recursively by $a_1 = s$ and

$$a_{n+1} = 2n + a_n \qquad (\text{for } n = 1, 2, \dots).$$

Prove that if the numbers $a_1, a_2, \dots, a_m$ are prime, then a_{s-1} is also prime.

The key idea is to prove by induction that a_n is in fact prime for all $n = 1, 2, \dots, s-1$.

Consider the minimal positive integer $t \ge 1 + m$ such that a_t is not prime. Then the numbers a_{2-t}, a_{3-t}, $\dots$, a_{-1}, a_0, a_1, $\dots$, a_{t-1} are all prime (where we permit negative indices). Let p denote the smallest prime factor of a_t; then

$$p < \sqrt{a_t} = \sqrt{t^2 - t + k} \le \sqrt{t^2 - t + 3m^2} < \sqrt{t^2 - t + 3t(t-1)} < 2t - 1.$$

So $p \le 2t - 2$, meaning we can find an integer $c \in [2 - t, t - 1]$ such that $c \equiv t \pmod p$. Then $a_c \equiv a_t \equiv 0 \pmod p$. As a_c is prime by assumption, this means $p = a_c \ge s$. Consequently $a_t^2 \ge p^2 \ge s^2$, whence $t \ge s$ as required.