OTIS Application

Year VI — OTIS 2020-2021

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Due: August 1, 2020 at 11:59PM PT

§1 Philosophy: this is not a test

The application problems are treated differently from what you may be used to. Many programs have an "entrance exam" or "qualifying test" or similar, which serve as an exam, and the highest scores get in. This is *not* the intention of the OTIS application; I already have your contest scores (and don't think meritocracy is needed here anyways).

Instead, the OTIS application problems are meant mostly for a different purpose:

- It implicitly serves as your summer reading. Part I of my geometry book [3] is basically a pre-requisite for OTIS, so working through the geometry problems in section A will check that you actually know the background.
- It lets me see your writing. If I have an easy time reading and understanding your solutions, that's usually a good sign that OTIS will work well. (This means it's to your advantage to write up solutions well!)
- It gives you some practice asking for help (see item 3 in instructions).
- It helps serve as a sanity check that you will have enough time to work on OTIS during the year. This packet consists of actual olympiad-level problems, so you can see what you are getting yourself into.

So, **please treat this like homework rather than a test**. In particular, you can even ask me for help on the problems (see item 3 in instructions). I will not just grade out of 7 and sum the scores (in fact, I probably won't even bother assigning scores). Instead, I am looking to see whether you are someone who is willing and able to solve olympiad problems and take the time to write them up cleanly.

Don't be discouraged if you find the problems challenging! If you start early, work diligently, and are willing to ask for hints, then I think you're likely to do very well.

§2 Instructions on solving

If you attended OTIS or MOP in any previous year, you do not have to do the problems. Skip to the section about submitting (and ignore the PDF upload step).

1. Try to solve as many problems as you can.
2. **You must write the solutions yourself**; don't copy-paste someone else's work.: If you have seen a problem before, you may write any solution you remember.

3. **You can ask me for help if you're stuck on something!** Just send me an email telling me what you've tried, and I'll try to push you in the right direction.[1] This is how OTIS works for admitted students, so why not practice now?

   For the geometry problems from [3], you can also use the hints in the back.

4. You can also use any other online or print references, e.g. searching the web. You may also ask other people for aid. However, I ask that you **reference any "outside sources"** that you used, for each problem, other than those in item 3.

5. Try to write your solutions clearly and completely; this matters significantly for decisions. See [2] for some suggestions.

§3 Instructions on submitting (before standard deadline)

6. Until August 1, 2020 at 11:59PM PT, applications may be submitted at:

   `https://forms.gle/LKqrGpR1gqY4yCVT9`

   This form will also contain some questions for you to fill out, such as your grade level and past experience.

7. Solutions to the problems should be submitted as a **single PDF**, which is not to exceed **10 megabytes**. Scans might exceed that limit, so I strongly suggest you use LaTeX instead.

8. The standard deadline is **August 1, 2020 at 11:59PM PT**. You should try to apply by this date if at all possible.

§4 Late applications

9. The window for late applications is **August 22, 2020 to February 28, 2021**.

10. **Only the Correspondence format** is available for late applications.

11. Late applications should be submitted by email instead of the form above. The email should be sent to `evan@evanchen.cc` and include:
    - Full name and grade level
    - Country (and state if you are from USA)
    - Past contest results and history (e.g. past USA(J)MO scores)
    - Statement of purpose: say a couple sentences about about yourself, why you want to do OTIS, and what you're aiming for this year.
    - A single PDF attachment with your solutions, at most 10 megabytse.

12. Late applications are processed on a rolling basis. Moreover, late applications are always billed for both semesters.

[1] Warning: I travel for MOP/IMO over the summer, so responses then will be slower. Start early.

Problems

§A Geometry

- **Reading**: It's suggested to read Part I of my book ([3]) as the material there is necessary (and sufficient) to solve these problems.
- All problems are themselves from EGMO [3], and you can use the hints there to help you get un-stuck.
- It is not essential that you typeset diagrams for these problems.
- All problems admit synthetic solutions, but computational approaches are okay too. Do whatever you need to.

Problem A.1 (#1.40, Canada 1991)**.** Let P be a point inside circle ω. Consider chords of ω passing through P. Prove that the midpoints of these chords all lie on a fixed circle.

Problem A.2 (#1.50, IMO 2013)**.** Let ABC be an acute triangle with orthocenter H, and let W be a point on the side $\overline{BC}$, between B and C. The points M and N are the feet of the altitudes drawn from B and C, respectively. Suppose ω_1 is the circumcircle of triangle BWN and X is a point such that $\overline{WX}$ is a diameter of ω_1. Similarly, ω_2 is the circumcircle of triangle CWM and Y is a point such that $\overline{WY}$ is a diameter of ω_2. Show that the points X, Y, and H are collinear.

Problem A.3 (#2.28, JMO 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Problem A.4 (#2.35, IMO 2009)**.** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

Problem A.5 (#3.25, USAMO 1993)**.** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

Problem A.6 (#3.30, USAMO 2015)**.** Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Problem A.7 (#4.47, USAMO 2011)**.** Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \qquad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \qquad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

Problem A.8 (#4.48, Japan 2009)**.** Triangle ABC has circumcircle Γ. A circle with center O is tangent to BC at P and internally to Γ at Q, so that Q lies on arc BC of Γ not containing A. Prove that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

Problem A.9 (#4.53, SL 2002)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let $\overline{AD}$ be an altitude of triangle ABC, and let M be the midpoint of $\overline{AD}$. If N is the common point of the circle Ω and $\overline{KM}$ (distinct from K), then prove Ω and the circumcircle of triangle BCN are tangent to each other.

§B Inequalities

- **Reading**: You should read *A Brief Introduction to Olympiad Inequalities* [1] as the material there is necessary (and sufficient) to solve these problems.

Problem B.1. Let a, b, c be positive reals. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \ge a + b + c.$$

Problem B.2. Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

Problem B.3. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

Problem B.4. Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

Problem B.5. For positive real numbers a, b, c satisfying $abc = 1$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}.$$

§C More

Problem C.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(xf(x) + f(y)) = f(x)^2 + y$$

holds for all real numbers x and y.

Problem C.2. Let a, b, c, d be real numbers such that $b - d \ge 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Problem C.3. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

Problem C.4 (Implies IMO 2017/3)**.** Let $d \ge 1$ be a real number. A hunter and an invisible rabbit play a game in the plane. The rabbit and hunter start at points A_0 and B_0, which are a distance d apart; the starting locations are known to both players. In the nth round of the game ($n \ge 1$), three things occur in order:

(i) The rabbit moves invisibly from A_{n-1} to a point A_n such that $A_{n-1}A_n = 1$.

(ii) The hunter has a tracking device (e.g. dog) which reports an approximate location P_n of the rabbit, such that $P_nA_n \le 1$.

(iii) The hunter moves visibly from B_{n-1} to a point B_n such that $B_{n-1}B_n = 1$.

Prove that after $\lfloor 500d \rfloor$ turns, the hunter cannot guarantee being within distance $\sqrt{d^2 + 1/2}$ from the rabbit.

References

[1] Evan Chen. A brief introduction to olympiad inequalities. http://web.evanchen.cc/handouts/Ineq/en.pdf.

[2] Evan Chen. Remarks on English. http://web.evanchen.cc/handouts/english/english.pdf.

[3] Evan Chen. *Euclidean geometry in mathematical olympiads*. MAA Problem Books Series. Mathematical Association of America, Washington, DC, 2016.