OTIS Application Homework

Year IX — OTIS 2023-2024

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Due: August 1, 2023, at 11:59PM PT

§1 Philosophy: this is not a test

The application problems are treated differently from what you may be used to. Many programs have an "entrance exam" or "qualifying test" or similar, which serve as an exam, and the highest scores get in. This is *not* the intention of the OTIS application; which is not meant to compare applicants. I like the FAQ from the Euler Circle, which I've adapted for OTIS:

> The traditional way admissions are competitive is there are N applicants and $M \ll N$ spaces available. That is NOT the case for OTIS, where $M > N$ at present. We dislike the idea of depriving students of an education; we will certainly not take any pride in doing so, and will never release statistics.

Instead, the OTIS application problems are meant mostly for a different purpose:

- It implicitly serves as your summer reading. Part I of my geometry book is basically a pre-requisite for OTIS, so working through the geometry problems in section A will check that you actually know the background.
- It lets me see your writing. If I have an easy time reading and understanding your solutions, that's usually a good sign that OTIS will work well. (This means it's to your advantage to write up solutions well!)
- It gives you some practice asking for help (see item 4 in instructions).
- It helps serve as a sanity check that you will have enough time to work on OTIS during the year. This packet consists of actual olympiad-level problems, so you can see what you are getting yourself into.

So, **please treat this like homework rather than a test**. In particular, you can even ask me for help on the problems (see item 4 in instructions). I will not just grade out of 7 and sum the scores (in fact, I probably won't even bother assigning scores). Instead, I am looking to see whether you are someone who is willing and able to solve olympiad problems and take the time to write them up cleanly.

Don't be discouraged if you find the problems challenging! If you start early, work diligently, and are willing to ask for hints, then I think you're likely to do well.

§2 Instructions on solving

1. **Returners**: If you attended OTIS or MOP in any previous year, you do not have to do the problems. Skip to the next section (and ignore the PDF upload).

   Newcomers: Try to solve as many problems as you can.

2. **You must write the solutions yourself**; don't copy-paste someone else's work.

3. If you have seen a problem before, you may write any solution you remember, or use any solution that you've written yourself in the past.

4. **You can ask me for help if you're stuck on something!** Just send me an email telling me what you've tried, and I'll try to push you in the right direction.[1] This is how OTIS works for admitted students, so why not practice now?

   For the geometry problems in EGMO, you can also use the hints in the back of the textbook.

5. You can also use any other online or print references, e.g. searching the web. You may also ask other people for aid. However, I ask that you **reference any "outside sources"** that you used, for each problem, other than those in item 4.

6. Try to write your solutions clearly and completely; this matters significantly for decisions. See `https://web.evanchen.cc/handouts/english/english.pdf` for some suggestions.

§3 Instructions on submitting before August 1, 2023

7. Until August 1, 2023, at 11:59PM PT, applications may be submitted at:

   `https://forms.gle/uCpRSiNow7GXj4if8`

   This form will also contain some questions for you to fill out, such as your grade level and background.

8. Solutions to the problems should be submitted as a **single PDF**, which is not to exceed **10 megabytes**. Scans might exceed that limit, so LaTeX is encouraged.

9. If you require financial aid, you must also fill out the separate form `https://forms.gle/jPmWaqAvAavKSAkYA`. No need to submit this form if you don't require financial aid.

10. The standard deadline is **August 1, 2023, at 11:59PM PT**. You should try to apply by this date if at all possible.

§4 Instructions on submitting before April 30, 2024

11. You can submit late applications between up until April 30, 2024. Note that you will not be paired with an instructor, though. (If you send a late application before September 1, 2023, it could take a while to process because August is a busy month for OTIS.)

12. The cost is still $240 per semester; joining late does not decrease the cost.

[1] Warning: I travel for MOP/IMO over the summer, so responses then will be slower. Start early.

13. Late applications should be submitted by email instead of the form above. The email should be sent to `evan@evanchen.cc` and include:
    - Full name and grade level
    - Name of your school
    - Country (and state if you are from USA)
    - Optionally, AoPS username or Discord username
    - Past contest results and history (e.g. past contest scores)
    - Statement of purpose: say a couple sentences about yourself, why you want to do OTIS, and what you're aiming for this year.
    - A single PDF attachment with your solutions, at most 10 megabytes.
    - Financial aid requests, if any. In this case, please also fill out `https://forms.gle/jPmWaqAvAavKSAkYA` in addition to stating this on the email.
    - Your answers to the reading comprehension test.

§5 Meme for your amusement

Homework problems

§A Geometry homework problems from my textbook

- **Reading**: It's suggested to read Part I of my book as the material there is necessary (and sufficient) to solve these problems.
- All problems are themselves from the first three chapters of my geometry book and you can use the hints provided.
- It is not essential that you typeset diagrams for these problems.
- All problems admit synthetic solutions, but computational approaches are okay too. Do whatever you need to.

Problem A.1 (#2.28, JMO 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Problem A.2 (#2.35, IMO 2009)**.** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

Problem A.3 (#3.25, USAMO 1993)**.** Let $ABCD$ be a quadrilateral whose diagonals are perpendicular and meet at E. Prove that the reflections of E across the sides of $ABCD$ are concyclic.

Problem A.4 (#3.30, USAMO 2015)**.** Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$.

As X varies on segment $\overline{PQ}$, show that M moves along a circle.

§B Inequalities homework problems

- **Reading**: You should read §2.1, §2.2, §2.4 of The OTIS Excerpts as the material there is necessary (and sufficient) to solve these problems.

Problem B.1. Suppose that $a^2 + b^2 + c^2 = 1$ for positive real numbers a, b, c. Find the minimum possible value of

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

Problem B.2. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+(a+b+c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

Problem B.3. Let a, b, c, d be positive reals with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \ge \frac{1}{3}.$$

§C Additional homework problems

Problem C.1 (Learn to code). Write a computer program to determine the number of ordered pairs of prime numbers (p,q) such that, when the number

$$N = p^2 + q^3$$

is written in decimal (without leading zeros), each digit from 0 to 9 appears exactly once.

You may use any programming language; if you don't have past coding experience, I recommend learning Python. Up to §4.4 of `https://docs.python.org/3/tutorial/index.html` should suffice. If you use C/C++, be careful of integer overflow.

Submit the numerical answer and the source code for the program (if using LaTeX, I suggest the `listings` package). As a confirmation, the correct answer is 10 times a prime number.

Problem C.2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$f(xf(x) + f(y)) = f(x)^2 + y$$

holds for all real numbers x and y.

Problem C.3. Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

Problem C.4. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. Ana wins if she is able to supply such a word, otherwise she loses. For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word. Which words can Ana pick so that she can win no matter what value of k Banana chooses?

§D Reading comprehension, to be submitted separately

This is designed to make sure you actually read the entire OTIS syllabus before you submit a complete application. Don't include these answers in your solution PDF.

- If you are applying at the normal time, enter your answers in the Google form.
- If you are applying late, include your answers in the email.

Problem D.1. What was the answer in the example outline?

Problem D.2. If you have

$$500\clubsuit + 300\heartsuit + 75\spadesuit + 40\diamondsuit$$

then what level are you?

Problem D.3. In the unit catalog, what is the name of the miscellaneous unit containing "the worst olympiad problems"? Your answer should be an eight-letter word.

§E Extra credit (for fun, not actually considered for admissions)

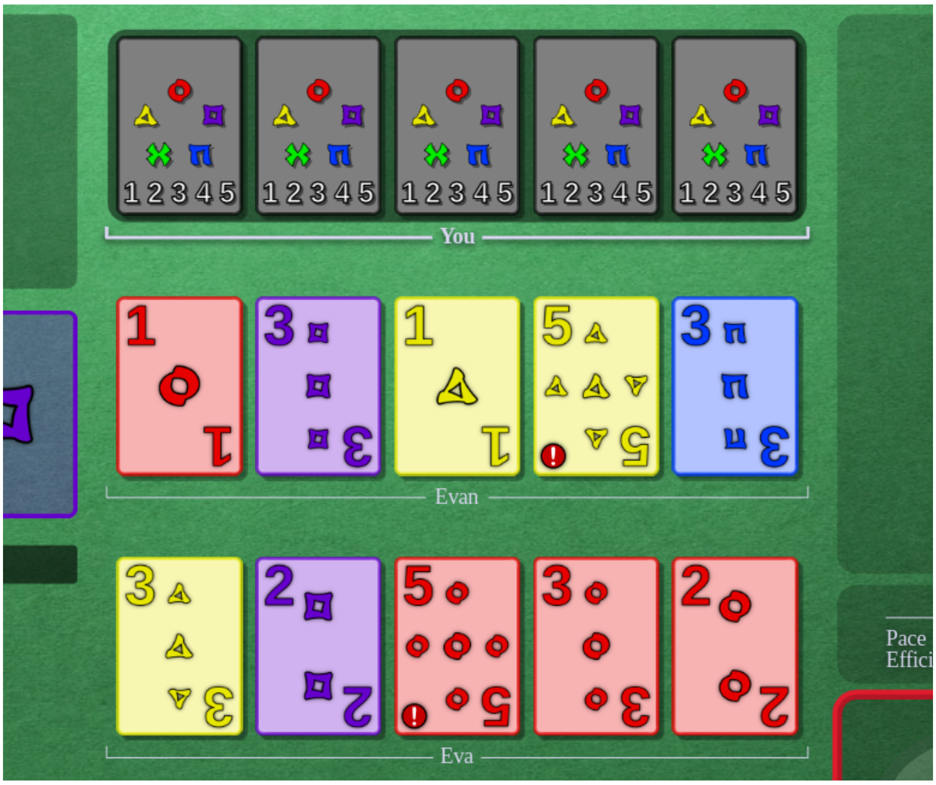
This section is only if you're really bored over the summer or something and are looking to pick up new hobbies.

Problem E.1. You're playing a beginner game of Hanabi with Evan and Eva using the conventions listed in https://tinyurl.com/hanabi-evan-intro. It's the first turn and the game looks as shown below.

(a) What are the legal moves under this convention set?

(b) Of these, which one do you think is the best and why?

If you play with higher-level conventions, you can answer for those instead.



Full image: https://web.evanchen.cc/upload/hanabi243.png

Problem E.2. Drop by https://mosp.evanchen.cc and solve as many of the puzzles as you can.

Note for the 2021 hunt, you'll want to have looked at the statements of the USAMO 2021 problems that year, since they are used in some of the puzzles. The 2021 hunt has solutions available, the 2022 hunt does not.

Problem E.3. Which Dreamcatcher song is the best?