Evan's PhD Notebook

https://github.com/vEnhance/evans-phd-notebook/

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§8.1 Background

Let F be a finite extension of $\mathbb{Q}_p$ for $p > 2$ and let E/F be an unramified quadratic field extension. Denote by ϖ a uniformizer of $\mathcal{O}_F$, such that $\bar{\varpi} = \varpi$, and let v be the associated valuation. Let η be the quadratic character attached to E/F by class field theory, so that $\eta(x) = -1^{v(x)}$.

§8.1.1 Symmetric space

We define the symmetric space

$$S_3(F) \coloneqq \{s \in \mathrm{GL}_3(E) \mid s\bar{s} = \mathrm{id}\}.$$

We also pay particular attention to the subspace which have $\mathcal{O}_E$ entries:

$$K_S \coloneqq S_3(F) \cap \mathrm{GL}_3(\mathcal{O}_E).$$

Lemma 8.1.1 (Cartan decomposition)
For each integer $m \geq 0$ let

$$K_{S,m} \coloneqq K_S \cdot \begin{bmatrix} 0 & 0 & \varpi^m \\ 0 & 1 & 0 \\ \varpi^{-m} & 0 & 0 \end{bmatrix}.$$

Then we have a decomposition

$$S_3(F) = \coprod_{m \geq 0} K_{S,m}.$$

For $r \geq 0$, define

$$\Omega_r \coloneqq S_3(F) \cap \varpi^{-m}\,\mathrm{GL}_3(\mathcal{O}_E).$$

We can re-parametrize the problem according to the following claim.

Claim 8.1.2 —

$$\Omega_r = K_{S,0} \sqcup K_{S,1} \sqcup \cdots \sqcup K_{S,r}.$$

If this claim is true (still need to check it), then an integral over each Ω_r lets us extract the integrals over $K_{S,m}$.

§8.1.2 Orbital integral

Define

$$H' \coloneqq \left\{ \begin{bmatrix} t_1 & t_2 \\ \bar{t}_2 & \bar{t}_1 \end{bmatrix} \right\} \cong \mathrm{GL}_2(F).$$

We embed H' into $\mathrm{GL}_3(F)$ by $h' \mapsto \left[\begin{smallmatrix} h' & 0 \\ 0 & 1 \end{smallmatrix}\right]$, which allows H to act on $\mathrm{GL}_3(F)$ and hence $S_3(F)$.

Now we can define the orbital integral.

Definition 8.1.3. For brevity let $\eta(h') := \eta(\det h')$ for $h' \in H'$. For $\gamma \in S_3(F)$ and $s \in \mathbb{C}$, we define the orbital integral by

$$O(\gamma, s) := \int_{g \in H'} \mathbf{1}_{\Omega_r}(\bar{g}^{-1}\gamma g)\eta(g) \left|\det(g)\right|_F^{-s} \, dg$$

where

$$dg = \kappa \cdot \frac{dt_1 \, dt_2}{|t_1\bar{t}_1 - t_2\bar{t}_2|_F^2}$$

for the constant $\kappa := (1-q^{-1})^{-1}(1-q^{-2})^{-1}$.

Indeed, for $h' \in H$ and $\gamma \in S_3(F)$ we have $h'\gamma(\bar{h}')^{-1} \in S_3(F)$ and so the indicator function is filtering based on which part of the Cartan decomposition that $h'\gamma(\bar{h}')^{-1}$ falls in.

Evidently $O(\gamma, s)$ only depends on the H'-orbit of γ. So it makes sense to pick a canonical representative for the H'-orbit to compute the orbital integral in terms of. For so-called *regular* γ, the representatives

$$\gamma(a,b,d) = \begin{bmatrix} a & 0 & 0 \\ b & -\bar{d} & 1 \\ c & 1-d\bar{d} & d \end{bmatrix} \in S_3(F); \quad \text{where } c = -a\bar{b} + bd$$

over all $a \in E^1$, $b \in E$, $d \in E$ for which $(1-d\bar{d})^2 - c\bar{c} \neq 0$, cover all the *regular* orbits, which are the ones we care about.

For $r = 0$, [Zha12] computes $\frac{\partial}{\partial s} O(\gamma, s)$ at $s = 0$ in terms of a, b, d. Our goal is to compute it for $r > 0$ too.

§8.2 Reparametrization in terms of valuations

§8.2.1 Computation of value in indicator function

We are integrating over $t_1 \in E$ and $t_2 \in E$. Regarding $g \in H'$ as an element of GL_3 as described before, we have

$$g = \begin{bmatrix} t_1 & t_2 & 0 \\ \bar{t}_2 & \bar{t}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We therefore have

$$\bar{g}^{-1} = \begin{bmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$\begin{aligned} \bar{g}^{-1}\gamma g &= \begin{bmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ b & -\bar{d} & 1 \\ c & 1-d\bar{d} & d \end{bmatrix} \begin{bmatrix} t_1 & t_2 & 0 \\ \bar{t}_2 & \bar{t}_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{t_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ \frac{-t_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \frac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} at_1 & at_2 & 0 \\ bt_1 - \bar{d}\bar{t}_2 & bt_2 - \bar{d}\bar{t}_1 & 1 \\ ct_1 + (1-d\bar{d})\bar{t}_2 & ct_2 + (1-d\bar{d})\bar{t}_1 & d \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \dfrac{at_1^2 - bt_1\bar{t}_2 + d\bar{t}_2^2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{at_1t_2 - bt_2\bar{t}_2 + d\bar{t}_1\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{-\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} \\ \dfrac{-at_1t_2 + bt_1\bar{t}_1 - d\bar{t}_1\bar{t}_2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{-at_2^2 + b\bar{t}_1t_2 - d\bar{t}_1^2}{t_1\bar{t}_1 - t_2\bar{t}_2} & \dfrac{\bar{t}_1}{t_1\bar{t}_1 - t_2\bar{t}_2} \\ ct_1 + (1 - d\bar{d})\bar{t}_2 & ct_2 + (1 - d\bar{d})\bar{t}_1 & d \end{bmatrix}$$

Let us define

$$t = t_2\bar{t}_1^{-1} \iff t_2 = t\bar{t}_1.$$

This lets us rewrite everything in terms of the ratio t and $t_1 \in E$:

$$\bar{g}^{-1}\gamma g = \begin{bmatrix} \dfrac{t_1^2(a - b\bar{t} + d\bar{t}^2)}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{t_1\bar{t}_1(at - bt\bar{t} + d\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{t_1 \cdot (-\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} \\ \dfrac{t_1\bar{t}_1(-at + b - d\bar{t})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{\bar{t}_1^2(-at^2 + bt - \bar{d})}{t_1\bar{t}_1(1 - t\bar{t})} & \dfrac{-\bar{t}_1}{t_1\bar{t}_1(1 - t\bar{t})} \\ t_1(c + (1 - d\bar{d})\bar{t}) & \bar{t}_1(ct + (1 - d\bar{d})) & d \end{bmatrix}$$

This new parametrization is better because t_1 only plays the role of a scale factor on the outside, with "interesting" terms only involving t. To make this further explicit, we write

$$t_1 = \varpi^{-m}\varepsilon$$

for $m \in \mathbb{Z}$ and $\varepsilon \in \mathcal{O}_E^\times$. Then we actually have

$$\begin{bmatrix} \bar{\varepsilon} & & \\ & \varepsilon & \\ & & 1 \end{bmatrix} \bar{g}^{-1}\gamma g \begin{bmatrix} \varepsilon^{-1} & & \\ & \bar{\varepsilon}^{-1} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} & \dfrac{at - bt\bar{t} + \bar{d}\bar{t}}{1 - t\bar{t}} & \dfrac{-\varpi^m\bar{t}}{1 - t\bar{t}} \\ \dfrac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} & \dfrac{-at^2 + bt - \bar{d}}{1 - t\bar{t}} & \dfrac{-\varpi^m}{1 - t\bar{t}} \\ \dfrac{c + (1 - d\bar{d})\bar{t}}{\varpi^m} & \dfrac{ct + (1 - d\bar{d})}{\varpi^m} & d \end{bmatrix}$$

For brevity, we will let $\Gamma(\gamma, t, m)$ denote the right-hand matrix. The conjugation by $\begin{bmatrix} \varepsilon^{-1} & & \\ & \bar{\varepsilon}^{-1} & \\ & & 1 \end{bmatrix}$ has no effect on any of the Ω_r, so that we can simply use

$$\mathbf{1}_{\Omega_r}(\bar{g}^{-1}\gamma g) = \mathbf{1}_{\Omega_r}(\Gamma(\gamma, t, m))$$

in the work that follows. By abuse of notation, we abbreviate

$$\mathbf{1}(\gamma, t, m) := \mathbf{1}_{\Omega_r}(\Gamma(\gamma, t, m)).$$

§8.2.2 Reparametrizing the integral in terms of t and m

From now on, following [Zha12] we always fix the notation

$$m = m(t_1) := -v(t_1)$$
$$n = n(t) := v(1 - t\bar{t}).$$

We need to rewrite the integral, phrased originally via $\mathrm{d}g$, in terms of the parameters t (hence n), m, and γ. We start by observing that

$$\det g = t_1\bar{t}_1 - t_2\bar{t}_2 = t_1\bar{t}_1(1 - t\bar{t})$$

which means that

$$v(\det g) = -2m + n$$

ergo

$$\begin{aligned} |\det g|_F &= q^{-v(\det g)} = q^{2m-n} \\ \eta(g) &= (-1)^{v(\det g)} = (-1)^n. \end{aligned}$$

Meanwhile, from $t_2 = t\bar{t}_1$ we derive

$$\mathrm{d}t_2 = |t_1|_E \, \mathrm{d}t = q^{2m} \, \mathrm{d}t.$$

Bringing this all into the orbital integral gives

$$\begin{aligned} O(\gamma, s) &= \kappa \int_{t,t_1 \in E} \mathbf{1}(\gamma, t, m)(-1)^n \left(q^{2m-n}\right)^{s-2} \mathrm{d}t_1 \cdot (q^{2m} \, \mathrm{d}t) \\ &= \kappa \int_{t,t_1 \in E} \mathbf{1}(\gamma, t, m)(-1)^n q^{s(2m-n)} \cdot q^{2n-2m} \, \mathrm{d}t \, \mathrm{d}t_1. \end{aligned}$$

§8.3 Setup

§8.3.1 Simplifying assumptions

For the purposes of [**Zha12**], we will only care about the following case:

Assumption 8.3.1

$$v\left((1 - d\bar{d})^2 - c\bar{c}\right) \equiv 1 \pmod 2$$

We will also assume:

Assumption 8.3.2
$v(d) \ge -r$.

This is fine because if this $v(d) < -r$ then the integral will always vanish (because the bottom-right entry of $\Gamma(\gamma, t, m)$ is no-good). Because of this, from (8.3) we then get

Corollary 8.3.3
$v(b) \ge -r$.

§8.3.2 Notations

As we described earlier, our goal is to give an answer in terms of

$$a \in E^1, \qquad b, d \in E, \qquad r > 0.$$

To simplify the notation in what follows, it will be convenient to define several quantities that reappear frequently. From Assumption 8.3.1, we may define

$$\delta := v(1 - d\bar{d}) = v(c) \ne -\infty. \tag{8.1}$$

Following [Zha12] we will also define

$$u := \frac{\bar{c}}{1 - d\bar{d}} \in \mathcal{O}_E^\times \tag{8.2}$$

so that $\nu(1 - u\bar{u}) \equiv 1 \pmod 2$ and

$$b = -au - \bar{d}\bar{u}. \tag{8.3}$$

Note that this gives us the following repeatedly used identity

$$b^2 - 4a\bar{d} = (au - \bar{d}\bar{u})^2 - 4a\bar{d}(1 - u\bar{u}). \tag{8.4}$$

Finally, define

$$\ell := v(b^2 - 4a\bar{d}). \tag{8.5}$$

We will also define one additional parameter useful when ℓ is even:

$$\lambda := v(1 - u\bar{u}) \equiv 1 \pmod 2. \tag{8.6}$$

In the case where ℓ is odd, we get (8.4) implying $\lambda = \ell$ and this definition will never be used — the orbital will be computed as a function of ℓ and δ (and r). However for even ℓ these numbers are never equal and our orbital integral will be stated in terms of ℓ, δ, and λ (and r).

§8.4 Description of the nonzero regions

§8.4.1 The case where $n \le 0$

Claim 8.4.1 — Whenever $n = 0$ (this requires $v(t) \ge 0$),

$$\mathbf{1}(\gamma, t, m) = \begin{cases} 1 & \text{if } -r \le m \le \delta + r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have to consider the nine entries of $\Gamma(\gamma, t, m)$ in tandem.

The upper 2×2 matrix is always in $\omega^{-r}\mathcal{O}_E$, because $v(t) \ge 0$, $v(d) \ge -r$, $v(b) \ge -r$, and $v(a) = 0$ suffices.

In the right column, since $v(t) \ge 0$ and $n = 0$, the condition is simply $m \ge -r$.

In the bottom row, we need $v\left(c + (1 - d\bar{d})\bar{t}\right) - m \ge -r$ and $v\left(ct + (1 - d\bar{d})\right) - m \ge -r$. If $v(t) > 0$ this is equivalent to $m - r \le \delta$. In the case where $v(t) = 0$ we instead use the observation that

$$\left[c + (1 - d\bar{d})\bar{t}\right] - \bar{t}\left[ct + (1 - d\bar{d})\right] = (1 - t\bar{t})c \tag{8.7}$$

which forces at least one of $ct + (1 - d\bar{d})$ and $c + (1 - d\bar{d})\bar{t}$ to have valuation δ. So the claim follows now. □

Claim 8.4.2 — Suppose $n = -2k < 0$, equivalently, $v(t) = -k < 0$, for some k.

$$\mathbf{1}(\gamma, t, m) = \begin{cases} 1 & \text{if } -r \le m + k \le \delta + r \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the previous claim, but simpler.

Since $k > 0$, the fraction $\frac{t^2}{1-t\bar{t}}$ has positive valuation, so the upper 2×2 of $\Gamma(\gamma, t, m)$ is always in $\varpi^{-r}\mathcal{O}_E$. Turning to the right column, the condition reads exactly $m + k \geq -r$. Finally, in the bottom row, from $v(t) > 0$ and $v(c) = \delta$ the condition is simply $-k + \delta - m \geq -r$. ☐

§8.4.2 Setup for $n > 0$

In this situation we evaluate over $n > 0$ only. In this case t is automatically a unit.

Consider the upper 2×2 matrix of $\Gamma(\gamma, t, m)$. Using the identities

$$\frac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} - \bar{t} \cdot \frac{at - bt\bar{t} + \bar{d}\bar{t}}{1 - t\bar{t}} = a - b\bar{t} \in \varpi^{-r}\mathcal{O}_E$$

$$\frac{a - b\bar{t} + \bar{d}\bar{t}^2}{1 - t\bar{t}} + \bar{t} \cdot \frac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} = a \in \varpi^{-r}\mathcal{O}_E$$

$$\frac{-at + b - \bar{d}\bar{t}}{1 - t\bar{t}} - \bar{t} \cdot \frac{-at^2 + bt - \bar{d}}{1 - t\bar{t}} = -a + b \in \varpi^{-r}\mathcal{O}_E,$$

it follows that as soon as one entry is in $\varpi^{-r}\mathcal{O}_E$, they all are. Meanwhile, the requirements on the other entries amount to

$$m \geq n - r \tag{8.8}$$

$$v\left(c + (1 - d\bar{d})\bar{t}\right) \geq m - r \tag{8.9}$$

$$v\left(ct + (1 - d\bar{d})\right) \geq m - r \tag{8.10}$$

According to the earlier identity (8.7), if (8.9) is assumed true, then (8.10) is equivalent to

$$\delta + v(1 - t\bar{t}) \geq m - r.$$

Meanwhile, since $v(c + (1 - d\bar{d})\bar{t}) = v(\bar{c} + (1 - d\bar{d})t)$, (8.9) is itself equivalent to

$$v(t + u) + \delta \geq m - r$$

by reading the definition of (8.2).

Finally, we use a tricky substitution

$$(2at - b)^2 - (b^2 - 4a\bar{d}) = -4a(-at^2 + bt - \bar{d})$$

to rewrite $v(-at^2 + bt - \bar{d}) \geq n - r$ as $v\left((2at - b)^2 - (b^2 - 4a\bar{d})\right) \geq n - r$.

In summary:

Claim 8.4.3 — Assume t is such that $n = v(1 - t\bar{t}) > 0$. Then $\mathbf{1}(\gamma, t, m) = 1$ if and only if

$$n - r \leq m \leq n + \delta + r$$

and t lies in the set specified by

$$v\left((2at - b)^2 - (b^2 - 4a\bar{d})\right) \geq n - r$$
$$v(t + u) \geq m - \delta - r.$$

§8.4.3 Volume lemma

The following two lemmas will be useful.

Lemma 8.4.4
Let $\xi \in \mathcal{O}_E^\times$ and let $n \geq 1$. Then the volume of the set

$$\{x \in E \mid v(1 - x\bar{x}) = n, \}$$

equals

$$q^{-n}(1 - q^{-2}).$$

Lemma 8.4.5 ([Zha12, Lemma 4.4])
Let $\xi \in \mathcal{O}_E^\times$ and let $n \geq \rho \geq 1$ be integers. Then the volume of the set

$$\{x \in E \mid v(1 - x\bar{x}) = n, \ v(x - \xi) \geq \rho\}$$

equals

$$\begin{cases} 0 & v(1 - \xi\bar{\xi}) < \rho \\ q^{-(n+\rho)}(1 - q^{-1}) & v(1 - \xi\bar{\xi}) \geq \rho. \end{cases}$$

We will also need to intersect two disks. In an ultrametric space, this is easy to do:

Lemma 8.4.6 (No MasterCard logo in an ultrametric space)
Choose $\xi_1, \xi_2 \in E$ and $\rho_1 \geq \rho_2 \geq 0$. Consider the two disks:

$$\begin{aligned} D_1 &= \{x \in E \mid v(x - \xi_1) \geq \rho_1\} \\ D_2 &= \{x \in E \mid v(x - \xi_2) \geq \rho_2\}. \end{aligned}$$

Then, if $v(\xi_1 - \xi_2) \geq \rho_2$, we have $D_1 \subseteq D_2$. If not, instead $D_1 \cap D_2 = \varnothing$.

Proof. Because E is an ultrametric space and $\mathrm{Vol}(D_1) \leq \mathrm{Vol}(D_2)$, we either have $D_1 \subseteq D_2$ or $D_1 \cap D_2 = \varnothing$. The latter condition checks which case we are in by testing if $\xi_1 \in D_2$, since $\xi_1 \in D_1$. $\square$

§8.4.4 The case where $n > 0$, and ℓ is odd

Considering $n > 0$ and $n - r \leq m \leq n + \delta + r$ as fixed, we compute the volume of the set of t for which $n = v(1 - t\bar{t})$ and $\mathbf{1}(\gamma, t, m) = 1$.

Supposing ℓ is odd, the condition

$$v\left((2at - b)^2 - (b^2 - 4a\bar{d})\right) \geq n - r$$

is equivalent to simultaneously the two conditions

$$v\left((2at - b)^2\right) \geq n - r \implies v\left(t - \frac{b}{2a}\right) \geq \left\lceil \frac{n - r}{2} \right\rceil \tag{8.11}$$

$$v(b^2 - 4a\bar{d}) \geq n - r \implies \ell \geq n - r. \tag{8.12}$$

We also had the requirement

$$v(t+u) \geq m - \delta - r. \tag{8.13}$$

Use Lemma 8.4.6 on (8.11) and (8.13), noting the distance between the two centers is exactly

$$v\left(u + \frac{b}{2a}\right) = v\left(\frac{au - \bar{d}\bar{u}}{2a}\right) = v(au - \bar{d}\bar{u}).$$

Considering that our disks have "radius" $\left\lceil \frac{n-r}{2} \right\rceil$ and $m - \delta - r$ respectively, we obtain two possible situations:

- If $m < \left\lceil \frac{n-r}{2} \right\rceil + \delta + r$ then Lemma 8.4.5 and Lemma 8.4.6 apply if and only if, respectively,

$$v(4 - b\bar{b}) \geq \left\lceil \frac{n-r}{2} \right\rceil \tag{8.14}$$

$$v(au - \bar{d}\bar{u}) \geq m - \delta - r. \tag{8.15}$$

- If $m \geq \left\lceil \frac{n-r}{2} \right\rceil + \delta + r$ then Lemma 8.4.5 and Lemma 8.4.6 apply if and only if, respectively,

$$v(1 - u\bar{u}) \geq m - \delta - r \tag{8.16}$$

$$v(au - \bar{d}\bar{u}) \geq \left\lceil \frac{n-r}{2} \right\rceil. \tag{8.17}$$

To proceed further, we need to prove a few properties. We list them in turn.

Fact 8.4.7. Whenever ℓ is odd, we must have

$$v(b) = v(d) = 0. \tag{8.18}$$

Proof of (8.18). If $v(d) \neq 0$, then $b = -au - \bar{d}\bar{u}$ is a unit, and hence so is $b^2 - 4a\bar{d}$, causing $\ell = 0$, contradiction. And if d is a unit, $\ell \neq 0$ means $v(b) = 0$ too. □

Next, note that (8.4) together with (8.18) and the assumption ℓ was odd implies

$$\ell = v(1 - u\bar{u}) < 2v(au - \bar{d}\bar{u}). \tag{8.19}$$

This implies that:

Fact 8.4.8. (8.15) and (8.17) are redundant for odd ℓ, i.e. they are automatically true whenever $n > 0$ and $n - r \leq m \leq n + \delta + r$.

Proof. Delete the ceilings. We have $\frac{n-r}{2} \leq \frac{\ell}{2} < v(au - \bar{d}\bar{u})$ in both cases. And in (8.14), we have $m - \delta - r \leq \frac{n-r}{2}$ anyway. □

Finally, the equation $v(4 - b\bar{b}) = -4au(1 - d\bar{d}) - \bar{b}(b^2 - 4a\bar{d})$ together with (8.18) implies

$$v(4 - b\bar{b}) \geq \min(\ell, \delta) \text{ with equality if } \ell \neq \delta. \tag{8.20}$$

Hence, a priori (8.20) suggests that we have a condition $n \leq r + 2\delta$ in addition to $n \leq r + \ell$. However, this condition also turns out to be redundant.

Lemma 8.4.9

When ℓ is odd we always have $\ell < 2\delta$.

Proof. To be written up. □

Putting all of this together, we find that the valid pairs (n, m) come in two cases.

¶ First case The first case is

$$\begin{aligned} 1 \le n &\le \ell + r, \\ n - r \le m &\le \left\lceil \frac{n-r}{2} \right\rceil + \delta + r - 1 \end{aligned} \tag{8.21}$$

where each (m, n) gives a volume contribution of

$$\begin{cases} q^{-n-\left\lceil \frac{n-r}{2} \right\rceil}\left(1-q^{-1}\right) & \text{if } n > r \\ q^{-n}\left(1-q^{-2}\right) & \text{if } n \le r. \end{cases}$$

¶ Second case The second case is

$$\begin{aligned} 1 \le n &\le \ell + r, \\ \max\left(n - r, \left\lceil \frac{n-r}{2} \right\rceil + \delta + r\right) \le m &\le \min(n, \ell) + \delta + r. \end{aligned} \tag{8.22}$$

where each (m, n) gives a volume contribution of

$$\begin{cases} q^{-n-(m-\delta-r)}\left(1-q^{-1}\right) & \text{if } m > \delta + r \\ q^{-n}\left(1-q^{-2}\right) & \text{if } m \le \delta + r. \end{cases}$$

Notice that $m \le \delta + r$ could only occur when $n \le r$.

§8.4.5 The case where $n > 0$, ℓ is even, $v(b) = v(d) = 0$

As before we consider $n > 0$ and $n - r \le m \le n + \delta + r$ as fixed, and seek to compute the volume of the set of t for which $n = v(1 - t\bar{t})$ and $\mathbf{1}(\gamma, t, m) = 0$.

Suppose ℓ is even. Then the left-hand side of (8.4) is a square, which we denote τ^2. In this case, we obtain

$$2v(\tau) = \ell = 2v(au - \bar{d}\bar{u}) > \lambda := v(1 - u\bar{u}).$$

Then the condition that

$$v\Big((2at - b)^2 - \underbrace{(b^2 - 4a\bar{d})}_{=\tau^2}\Big) \ge n - r$$

falls into three disjoint parts:

- Both $v\left((2at - b)^2\right) \ge n - r$ and $\ell = v(\tau^2) \ge n - r$ hold, as in the ℓ odd case.
- We have $\ell = v(\tau^2) < n - r$ (hence $v\left((2at - b)^2\right) < n - r$ too) but

  $$v(2at - b \mp \tau) \ge (n - r) - \ell/2 > 0$$

  which in particular implies $v(2at - b \pm \tau) = \ell/2$. This is two parts, corresponding to the choice of $\pm$.

We analyze the second case since the first case is the same as before (as we are assuming (8.18) in this section; it does not follow for ℓ even). The constrains on t become the two circles

$$v\left(t - \frac{b \pm \tau}{2a}\right) \ge n - \ell/2 - r \tag{8.23}$$

$$v(t+u) \ge m - \delta - r. \tag{8.24}$$

Note that

$$1 - \frac{b \pm \tau}{2a} \cdot \frac{\bar{b} \pm \bar{\tau}}{2\bar{a}} = \frac{4 - \mathrm{N}(b \pm \tau)}{4}$$

The distance between the two circles has valuation

$$v\left(u + \frac{b \pm \tau}{2}\right) = v(au - \bar{d}\bar{u} \pm \tau).$$

Since $(au - \bar{d}\bar{u})^2 - \tau^2 = 4a\bar{d}(1 - u\bar{u})$, we agree now to fix the choice of the square root of τ such that

$$v(au - \bar{d}\bar{u} + \tau) = \lambda - v(\tau) \quad \text{and} \quad v(au - \bar{d}\bar{u} - \tau) = v(\tau). \tag{8.25}$$

From $v(b) = v(d) = 0$ and (8.4), we have

$$\ell = 2v(\tau) = 2v(au - \bar{d}\bar{u}) < \lambda.$$

When $v(b) = v(d) = 0$ we also automatically have $\delta, \ell \ge 0$

This lets us invoke [**Zha12**, Lemma 4.7] to evaluate $v(4 - \mathrm{N}(b \pm \tau))$: we have

$$\lambda + \delta - \ell = v\left(4 - \mathrm{N}(b + \tau)\right)$$

$$\delta = v\left(4 - \mathrm{N}(b - \tau)\right).$$

So we obtain $2 \cdot 2 = 4$ total cases.

¶ **Case** 1^+**.** Suppose $m < n - \frac{\ell}{2} + \delta$, and we choose $\frac{b+\tau}{2a}$. Then the contribution is nonempty if and only if

$$\lambda + \delta - \ell = v(4 - \mathrm{N}(b+\tau)) \ge n - \frac{\ell}{2} - r$$

$$\lambda - \ell/2 = v(au - \bar{d}\bar{u} + \tau) \ge m - \delta - r.$$

Compiling all seven constraints gives that the valid pairs (m, n) are those for which

$$\max(1, \ell + r + 1) \le n \le -\frac{\ell}{2} + \delta + r + \lambda,$$

$$n - r \le m \le \min\left(n + \delta + r, n - \frac{\ell}{2} + \delta - 1, \lambda - \frac{\ell}{2} + \delta + r\right)$$

However, $n + \delta + r \ge n - \frac{\ell}{2} + r$ is clear. So this equation can be whittled down to

$$\begin{aligned} \max(1, \ell + r + 1) \le n &\le -\frac{\ell}{2} + \delta + r + \lambda, \\ n - r \le m &\le \min\left(n - \frac{\ell}{2} + \delta - 1, \lambda - \frac{\ell}{2} + \delta + r\right). \end{aligned} \tag{8.26}$$

Each (m, n) gives a volume contribution of

$$q^{-n-(n-\ell/2-r)}\left(1 - q^{-1}\right).$$

¶ Case 1^-. Suppose $m < n - \frac{\ell}{2} + \delta$, and we choose $\frac{b-\tau}{2a}$. Then the disks have nonempty intersection whenever

$$\begin{aligned} \delta = v(4 - \mathrm{N}(b-\tau)) &\geq n - \frac{\ell}{2} - r \\ \ell/2 = v(au - \bar{d}\bar{u} - \tau) &\geq m - \delta - r. \end{aligned}$$

Compiling all seven constraints gives that the valid pairs (m, n) are those for which

$$\begin{aligned} &\max(1, \ell + r + 1) \leq n \leq \frac{\ell}{2} + \delta + r, \\ &\qquad n - r \leq m \leq \min\left(n - \frac{\ell}{2} + \delta - 1, \frac{\ell}{2} + \delta + r, n + \delta + r\right) \end{aligned}$$

However, $n - \frac{\ell}{2} + \delta - 1 \geq \frac{\ell}{2} + \delta + r$ hold automatically once $n \geq \ell + r + 1$, and $n + \delta + r \geq n - \frac{\ell}{2} + \delta - 1$ is true for $\ell \geq 0$. So we can simplify this to

$$\begin{aligned} \max(1, \ell + r + 1) \leq n &\leq \frac{\ell}{2} + \delta + r, \\ n - r \leq m &\leq \frac{\ell}{2} + \delta + r. \end{aligned} \tag{8.27}$$

As in the previous case, (m, n) gives a volume contribution of

$$q^{-n-(n-\ell/2-r)}\left(1 - q^{-1}\right).$$

¶ Case 2^+. Suppose $m \geq n - \frac{\ell}{2} + \delta$, and we choose $\frac{b+\tau}{2a}$. Then the contribution is nonempty if and only if

$$\begin{aligned} \lambda &\geq m - \delta - r \\ \lambda - \ell/2 = v(au - \bar{d}\bar{u} + \tau) &\geq n - \frac{\ell}{2} - r. \end{aligned}$$

Rearranging gives that the valid pairs (m, n) are those for which

$$\begin{aligned} \max(1, \ell + r + 1) \leq n \leq \lambda + r \\ \max\left(n - r, n - \frac{\ell}{2} + \delta\right) \leq m \leq \min(n, \lambda) + r + \delta. \end{aligned} \tag{8.28}$$

Here, each (m, n) gives a volume contribution of

$$q^{-n-(m-\delta-r)}\left(1 - q^{-1}\right).$$

¶ Case 2^-. Suppose $m \geq n - \frac{\ell}{2} + \delta$, and we choose $\frac{b+\tau}{2a}$. Then the disks have nonempty intersection whenever

$$\begin{aligned} \lambda &\geq m - \delta - r \\ \ell/2 = v(au - \bar{d}\bar{u} - \tau) &\geq n - \frac{\ell}{2} - r. \end{aligned}$$

The latter inequality contradicts the assumption that $n > \ell + r$, so this case can never occur.

§8.5 Evaluation of the integral

§8.5.1 Region where $n \le 0$ for all values of ℓ

Proposition 8.5.1

The contribution to the integral $O(\gamma, s)$ over $n \le 0$ is exactly

$$I_{n\le 0} := q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} q^{-2js} = q^{-2rs} + \dots + q^{2(\delta+r)s}.$$

Proof. For $n = 0$ we get a contribution of

$$\begin{aligned}
&\kappa \int_{t,t_1\in E} \mathbf{1}(n=0)\mathbf{1}(\gamma,t,m) q^{2s\cdot m} q^{-2m}\, \mathrm{d}t\, \mathrm{d}t_1 \\
&= \kappa \operatorname{Vol}(t : n = 0) \sum_{m=-r}^{\delta+r} \operatorname{Vol}(t_1 : -v(t_1) = m) q^{2m(s-1)} \\
&= \kappa \left(1 - \frac{q+1}{q^2}\right) \sum_{m=-r}^{\delta+r} \left(q^{2m}\left(1-q^{-2}\right)\right) q^{2m(s-1)} \\
&= \kappa \left(1 - \frac{q+1}{q^2}\right) \left(1-q^{-2}\right) \sum_{m=-r}^{\delta+r} q^{2ms}.
\end{aligned}$$

For the region where $v(t) = -k < 0$, for each individual $k > 0$,

$$\begin{aligned}
&\kappa \int_{t,t_1\in E} \mathbf{1}(v(t)=-k)\mathbf{1}(\gamma,t,m) q^{s(2m-n)} q^{2n-2m}\, \mathrm{d}t\, \mathrm{d}t_1 \\
&= \kappa \operatorname{Vol}(t : v(t) = -k) \sum_{m=-r-k}^{\delta+r-k} \operatorname{Vol}(t_1 : -v(t_1) = m) q^{s(2m+2k)-4k-2m} \\
&= \kappa q^{2k} \left(1-q^{-2}\right) \sum_{m=-r-k}^{\delta+r-k} \left(q^{2m}\left(1-q^{-2}\right)\right) q^{s(2m+2k)-4k-2m} \\
&= \kappa q^{-2k} \left(1-q^{-2}\right)^2 \sum_{m=-r-k}^{\delta+r-k} q^{2(m+k)s} \\
&= \kappa q^{-2k} \left(1-q^{-2}\right)^2 \sum_{i=-r}^{\delta+r} q^{2is}.
\end{aligned}$$

Since $\sum_{k>0} q^{-2k} = \frac{q^{-2}}{1-q^{-2}}$, we find that the total contribution across both the $n = 0$ case and the $k > 0$ case is

$$\begin{aligned}
&\left(\left(1 - \frac{q+1}{q^2}\right)\left(1-q^{-2}\right) + q^{-2}(1-q^{-2})\right) \kappa \sum_{i=-r}^{\delta+r} q^{2is} \\
&= \left(1-q^{-1}\right)\left(1-q^{-2}\right) \kappa \sum_{i=-r}^{\delta+r} q^{2is} \\
&= \sum_{i=-r}^{\delta+r} q^{2is}.
\end{aligned}$$

This equals the claimed sum above. (We write it over $0 \le j \le \delta + 2r$ for consistency with a later part.) □

§8.5.2 Region where $n > 0$ for odd ℓ

Again using $\mathrm{Vol}(t_1 : -v(t_1) = m) = q^{2m}(1-q^{-2})$, summing all the cases gives

$$\begin{aligned}
I^{\text{odd}}_{n>0} &:= \kappa \int_{t,t_1 \in E} \mathbf{1}(n>0)\mathbf{1}(\gamma,t,m) \\
&= \kappa \sum_{n=1}^{r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} q^{-n}\left(1-q^{-2}\right) \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&\quad + \kappa \sum_{n=r+1}^{\ell+r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} q^{-n-\lceil \frac{n-r}{2} \rceil}\left(1-q^{-1}\right) \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&\quad + \kappa \sum_{n=1}^{r} \sum_{m=\max(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r)}^{\delta + r} q^{-n}\left(1-q^{-2}\right) \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&\quad + \kappa \sum_{n=1}^{\ell+r} \sum_{m=\max(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r, \delta + r + 1)}^{\min(n,\ell)+\delta+r} q^{-n-(m-\delta-r)}\left(1-q^{-1}\right) \\
&\qquad \cdot \left((-1)^n q^{s(2m-n)} q^{2n-2m}\right)\left(q^{2m}(1-q^{-2})\right) \\
&= \sum_{n=1}^{r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=r+1}^{\ell+r} \sum_{m=n-r}^{\lceil \frac{n-r}{2} \rceil + \delta + r - 1} q^{\lfloor \frac{n+r}{2} \rfloor} \cdot (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=1}^{r} \sum_{m=\max(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r)}^{\delta + r} q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(2m-n)} \\
&\quad + \sum_{n=1}^{\ell+r} \sum_{m=\max(n-r, \lceil \frac{n-r}{2} \rceil + \delta + r, \delta + r + 1)}^{\min(n,\ell)+\delta+r} q^{n-(m-\delta-r)} \cdot (-1)^n q^{s(2m-n)}.
\end{aligned}$$

To simplify the expressions, we replace the summation variable m with

$$j := (n + \delta + r) - m \ge 0.$$

In that case,

$$2m - n = 2(\delta + n + r - j) - n = n + 2\delta + 2r - 2j.$$

Then the expression rewrites as

$$\begin{aligned}
I^{\text{odd}}_{n>0} &= \sum_{n=1}^{r} \sum_{j=\lfloor \frac{n+r}{2} \rfloor + 1}^{\delta + 2r} q^{n}\left(1+q^{-1}\right) \cdot (-1)^n q^{s(n+2\delta+2r-2j)} \\
&\quad + \sum_{n=r+1}^{\ell+r} \sum_{j=\lfloor \frac{n+r}{2} \rfloor + 1}^{\delta + 2r} q^{\lfloor \frac{n+r}{2} \rfloor} \cdot (-1)^n q^{s(n+2\delta+2r-2j)}
\end{aligned}$$

$$+\sum_{n=1}^{r}\sum_{j=n}^{\min(\delta+2r,\lfloor\frac{n+r}{2}\rfloor)} q^n\left(1+q^{-1}\right)\cdot(-1)^n q^{s(n+2\delta+2r-2j)}$$
$$+\sum_{n=1}^{\ell+r}\sum_{j=\max(0,n-\ell)}^{\min(\delta+2r,\lfloor\frac{n+r}{2}\rfloor,n-1)} q^j\cdot(-1)^n q^{s(n+2\delta+2r-2j)}.$$

We interchange the order of summation so that it is first over j and then n. There are four double sums to interchange.

- The first double sum runs from $j=\left\lfloor\frac{r+1}{2}\right\rfloor+1$ to $j=\delta+2r$. In addition to $1\le n\le r$, we need $\left\lfloor\frac{n+r}{2}\right\rfloor+1\le j$, which solves to $\frac{n+r}{2}\le j-\frac{1}{2}$ or $n\le 2j-1-r$. Thus the condition on n is
$$1\le n\le\min(2j-1-r,r).$$
- The second double sum runs from $j=r+1$ to $\delta+2r$. We also need $r+1\le n\le\ell+r$ and $n\le 2j-1-r$. Hence, the desired condition on n is
$$r+1\le n\le\min(2j-1-r,\ell+r).$$
- The third double runs from $j=1$ to $j=r$. Meanwhile, the values of n need to satisfy $1\le n\le r$, $n\le j$ and $j\le\left\lfloor\frac{n+r}{2}\right\rfloor \implies n\ge 2j-r$, consequently we just obtain
$$\max(1,2j-r)\le n\le j.$$
- The fourth double sum runs $j=0$ to
$$j=\min\left(\delta+2r,\left\lfloor\frac{\ell}{2}\right\rfloor+r,\ell+r-1\right)=\left\lfloor\frac{\ell}{2}\right\rfloor+r$$
again because of Lemma 8.4.9. Meanwhile, we require $1\le n\le\ell+r$, $j\ge n-\ell$, $j\le n-1$, as well as $j\le\left\lfloor\frac{n+r}{2}\right\rfloor \iff n\ge 2j-r$. Putting these four conditions together gives
$$\max(j+1,2j-r)\le n\le\ell+\min(j,r).$$

Hence we get

$$I^{\text{odd}}_{n>0}=\sum_{j=\lfloor\frac{r+1}{2}\rfloor+1}^{\delta+2r}\sum_{n=1}^{\min(2j-1-r,r)} q^n\left(1+q^{-1}\right)\cdot(-1)^n q^{s(n+2\delta+2r-2j)}$$
$$+\sum_{j=r+1}^{\delta+2r}\sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\lfloor\frac{n+r}{2}\rfloor}\cdot(-1)^n q^{s(n+2\delta+2r-2j)}$$
$$+\sum_{j=1}^{r}\sum_{n=\max(1,2j-r)}^{j} q^n\left(1+q^{-1}\right)\cdot(-1)^n q^{s(n+2\delta+2r-2j)}$$
$$+\sum_{j=0}^{\lfloor\frac{\ell}{2}\rfloor+r}\sum_{n=\max(j+1,2j-r)}^{\ell+\min(j,r)} q^j\cdot(-1)^n q^{s(n+2\delta+2r-2j)}.$$

At this point, we can unify the sum over j by noting that for j outside of the summation range, the inner sum is empty anyway. Specifically, note that:

- In the first and second double sum, the inner sum over n is empty anyway when $j < r$.
- In the third double sum, adding $j = 0$ does not introduce new terms. Moreover, when $j > r$ the inner sum over n is also empty anyway.
- In the fourth double sum, if $j > \left\lfloor \frac{\ell}{2} \right\rfloor + r$, the inner double sum vanishes since $2j - r > \ell + \min(j, r)$ in that case.

So we can unify all four double sums to run over $0 \le j \le \delta + 2r$, simplifying the expression to just

$$\begin{aligned} I^{\text{odd}}_{n>0} = q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} \Bigg(& \sum_{n=1}^{\min(2j-1-r,r)} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\left\lfloor \frac{n+r}{2} \right\rfloor} \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=\max(1,2j-r)}^{j} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=\max(j+1,2j-r)}^{\ell+\min(j,r)} q^j \cdot (-1)^n q^{s(n-2j)} \Bigg). \end{aligned}$$

§8.5.3 Completed case when ℓ is odd

Combining the previous two results gives

$$\begin{aligned} I_{n\le 0} + I^{\text{odd}}_{n>0} = q^{2(\delta+r)s} \sum_{j=0}^{\delta+2r} \Bigg(& q^{-2js} + \sum_{n=1}^{\min(2j-1-r,r)} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=r+1}^{\min(2j-1-r,\ell+r)} q^{\left\lfloor \frac{n+r}{2} \right\rfloor} \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=\max(1,2j-r)}^{j} q^n \left(1+q^{-1}\right) \cdot (-1)^n q^{s(n-2j)} \\ & + \sum_{n=\max(j+1,2j-r)}^{\ell+\min(j,r)} q^j \cdot (-1)^n q^{s(n-2j)} \Bigg). \end{aligned}$$

§8.5.4 Region where $n > 0$ for even ℓ

References

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