A Journey to the IMO

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The International Mathematics Olympiad (IMO) is undoubtedly the most prestigious math competition that exists for high-school students. What I found strange in my own IMO experience was that even though there were a lot of dedicated and talented aspirants worldwide, at the main event itself, about 25% contestants could not even solve a single problem. Upon reflecting about my own journey, I realised that this came down to 3 reasons:

1. A poor understanding of the IMO's social demands.
2. A poor understanding of the IMO's mathematical demands.
3. Sub-optimal preparation strategies.

Unfortunately, there isn't much guidance available online to help resolve these issues, especially for contestants from countries without strong training programmes. I hope to fill this vacuum with this document.

In section 1, I address issue 1 by outlining my own journey to the IMO. The reader should think about what learnings can be extrapolated from these experiences, which of those learnings can be applied to them, and how. In section 2, I address issue 2 by working through an IMO problem. In section 3, I address issue 3 by discussing the different approaches to preparation, and how one can create an optimal preparation strategy for themselves. Finally, I end with some key takeaways (my own!).

1 My Journey to the IMO

I belong to Nepal, a small developing country that is relatively new to the IMO (we only started participating in 2017). The competition is largely unheard of in schools here; the selection tests aren't as difficult to crack; and the training camps are still in their infancy. Quite a lot of countries participating in the IMO are in a similar state. Had I been from India, China or the US, my journey would've been radically different, arguably even much harder. Hence I urge the reader to think about how everything being said in this document applies to them.

My story begins in early 2020, when my general enthusiasm about math took a very competitive turn, and I became invested in making it to the IMO team from Nepal. We contacted the organisers and learnt about the stages of selection. At the time of writing, there are 5 rounds of selection in Nepal, which I'm listing here for context:

1. **Top 100 selection:** Tests of basic mathematics are conducted in each district of the country. The top 100 participants here advance to the next round.
2. **Top 50 selection:** After some sessions discussing basic topics of Olympiad Mathematics (such as Angle Chase), the top 50 selection test is conducted. The problems are relatively simple, yet olympiad-style.
3. **Top 25 selection:** This test is conducted after about a month after Round 2. The problems become harder and more complex here.
4. **Team Selection Test (TST):** After about one more month, the TST is conducted to select the final 6 contestants to represent Nepal at the IMO.
5. **Interview:** Sometimes, the scores at the TST become close such that it would be hard to clearly distinguish who gets selected. To resolve this, interviews are conducted. Based on this, they also select the backup contestants (to replace in case someone from the team needs to back out).

Thankfully, I was early: when I started, I was just about to finish grade 9. This meant that I had 3 strikes at the IMO - a lot more time and opportunities than most others.

1.1 Strike 1

Since I studied in a boarding school in India, coming back to give so many exams would've been difficult. So we contacted the organisers, and seeing a basic level of competence at math, they decided to make an exception and directly qualify me to Round 2.

So far, I knew absolutely nothing about olympiad math and was overly reliant on the training camps. Weeks went by, and I just didn't realise that I had to take charge of my preparation: there's too much content to cover in formal training sessions, and being new (and suddenly online), our organisers really weren't that sure about how to conduct them.

And so with the little knowledge that I had, and the reference of a long list of syllabus on the website, I started preparing through random websites online. A few weeks later, I got to know the Top 50 selections were about to happen. After struggling for 3 hours on 8 problems, I had solved a few of them, and that was enough to get me through. And so, my preparation continued: sincerely, but somewhat cluelessly. I just did whatever felt appropriate. Then came the Top 25 selections, which too I made through. Seeing my chances, I stepped up my preparation, spending over 2 hours on this each day. I knew I was low on time. A day before the TST, I was flipping my way through *Challenge and Thrill of Pre-college Mathematics* in an attempt to memorise all the theorems and results. I didn't solve a single problem; just learnt up all the theory (only to forget it all 2 days later). How ridiculous! Still, I solved managed to solve some of the problems, and ranked 10th there. That may sound good, but they only take 6 participants for the IMO.

I personally wasn't very disappointed by this result. I knew I wasn't prepared enough. Secretly, I was even glad that it happened. Had I been selected, I would likely have returned with 0 points. I personally find that shameful: it suggests that you made it *because* of the country where you belong, not in spite of it.

1.2 Strike 2

The experience I gained from Strike 1 was invaluable in that it fundamentally changed my approach to olympiad mathematics. I figured out that preparation had to be deep, broad and comprehensive. And this was impossible without consistent effort and a structured preparation scheme.

Based on my good performance in Strike 1, the organisers decided to directly select me for round 4, the Team Selection Test. I correctly solved 2 out of the 4 problems. Under usual circumstances, I knew this wouldn't be good enough. So, I tried really hard to solve another of the remaining two problems - but to no avail. Luckily, not many people solved two or more problems, and ranking 5th, I got selected!

I now had about 2 months to prepare for the IMO, and it didn't take me long to realise how little that was. To ensure that I was using my time optimally, I reached out to former IMO medalists and coaches for advice of all kinds - book suggestions, websites, preparation tips, understanding the relative importance of things, etc. Based on all this advice, I drafted a reasonable week-wise study plan. It certainly wasn't easy, but I think it paid off.

Before long, it was July and the IMO was about to begin. By now, even my 'reasonable' study plan proved to be unreasonable and I hadn't covered all the topics. Most worryingly, I didn't know much about Number Theory or Inequalities. I was left with no choice but to do the same kind of last minute scramble as in Strike 1. Still, I was confident I knew Combinatorics, Geometry and Functional Equations well. Although the initial target of winning a gold medal was clearly out of reach, I was still hopeful for bronze or silver.

Soon, the first big day was here – Day 1 of the IMO. The pandemic and lockdowns had completely stripped the glamour of the international event out. Instead of a trip to St. Petersburg, Russia, here we were sitting in a restaurant-turned-exam-center in Kathmandu, getting ready for what would be an exhausting day. Indeed, the problems were really challenging, but I had a strategy. A former medalist had suggested to me that no matter how much time it took, I first *completely solve* Problem 1 and only then look at the other two (harder) problems. It made sense to me and I stuck by this principle. Sure enough, I found myself making progress – slowly, but progressing. I took me 3 hours, but I did it. To my disappointment, that was all I managed for the day. Problem 2 was a strange inequality, and problem 3 was a really complicated geometry problem. I left the hall knowing that I was at least getting an award - hopefully a medal.

The second day was a disaster. I accidentally drank some coffee the night before, and that was enough to render me sleepless until 3AM. Naturally, my brain wasn't functioning at full capacity the next day. Even then, I made the blunder of neglecting the advice that had worked wonders just a day before: I got carried away with problem 5 (combinatorics), to no avail. By the time I gave a sincere attempt at problem 4 (geometry), I was both running out of time and brainpower. I ended up drafting solutions that I knew were mostly rubbish, in the hopes of getting some partial points. In the end, I got none. I was disappointed, but nonetheless glad that I didn't return with 0 points. In the end, I got the Honorable Mention, exactly as expected.

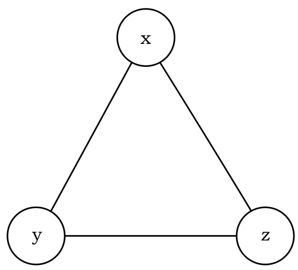
2 The IMO Problem 1

In order to give you a flavour of what IMO problems are like (and what you would prepare for), I would like to take you through my thinking process for the one IMO problem I did solve. I'll then formulate my approach into a proper solution. Unlike most other IMO problems, this one does not require much theoretical knowledge, and is thus the perfect example.

The problem (IMO 2021 P1): Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, ..., 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

My approach: We first want to find out the situation when the problem statement holds. A little bit of experimentation leads us to thinking: "If we have 3 cards, any two of which add up to a perfect square, then we should be done." This is because Ivan has to split them into 2 piles, and when splitting 3 cards into 2 piles, at least one pile will have 2 cards (pigeonhole principle). But by our choice of the cards, these two would sum up to a perfect square.

To arrive at this observation, I was thinking in terms of graphs, where the cards are vertices and the edges denote that the cards sum to a perfect square. Even though this isn't really necessary, I somehow found it easier to relate to.



Let's call these 3 cards x, y and z. We may assume $x < y < z$ without any loss of generality. So we have

$$x + y = ()^2$$
$$x + z = ()^2$$
$$y + z = ()^2$$

Experimenting with the values we obtain from the case $n = 100$, I worked out 2 further observations:

1. The squares we obtained would best be consecutive squares.
2. The three squares have to be two $(odd)^2$ and one $(even)^2$. We can check that no other configuration satisfies the parity constraints of the equations.

With these insights, we reframe our equations as follows:

$$x + y = (2k - 1)^2 \tag{1}$$
$$x + z = (2k)^2 \tag{2}$$
$$y + z = (2k + 1)^2 \tag{3}$$

where k is some positive integer.

But this is just a sequence of 3 linear equations and so we can easily solve for x, y and z in terms of k. We first add all 3 equations up:

$$\begin{aligned} 2x + 2y + 2z &= (2k-1)^2 + (2k)^2 + (2k+1)^2 \\ &= (4k^2 - 4k + 1) + (4k^2) + (4k^2 + 4k + 1) \\ &= 12k^2 + 2 \end{aligned}$$

Dividing by 2,

$$x + y + z = 6k^2 + 1 \tag{4}$$

Now we can subtract each of (3), (2), and (1) from (4) to obtain x, y and z respectively in terms of k.

$$x = (x + y + z) - (y + z) = (6k^2 + 1) - (4k^2 + 4k + 1) = 2k^2 - 4k$$
$$y = (x + y + z) - (x + z) = (6k^2 + 1) - (4k^2) = 2k^2 + 1$$
$$z = (x + y + z) - (x + y) = (6k^2 + 1) - (4k^2 - 4k + 1) = 2k^2 + 4k$$

Great. So now we know that "If we have the three cards $(2k^2 - 4k), (2k^2 + 1)$ and $(2k^2 + 4k)$ for some value of k, then the problem statement holds".

Now we need to show that these 3 cards do actually exist in the deck, for some integer k. This is basically the same as trying to show $n \leq x, y, z \leq 2n$ since they are obviously all integers. However, since $x < y < z$, we only need to show that $x = 2k^2 - 4k \geq n$ and $z = 2k^2 + 4k \leq 2n$ for some integer k. Observing that both equations have a similar right hand side, I realised we can combine them to obtain

$$k^2 + 2k \leq n \leq 2k^2 - 4k$$

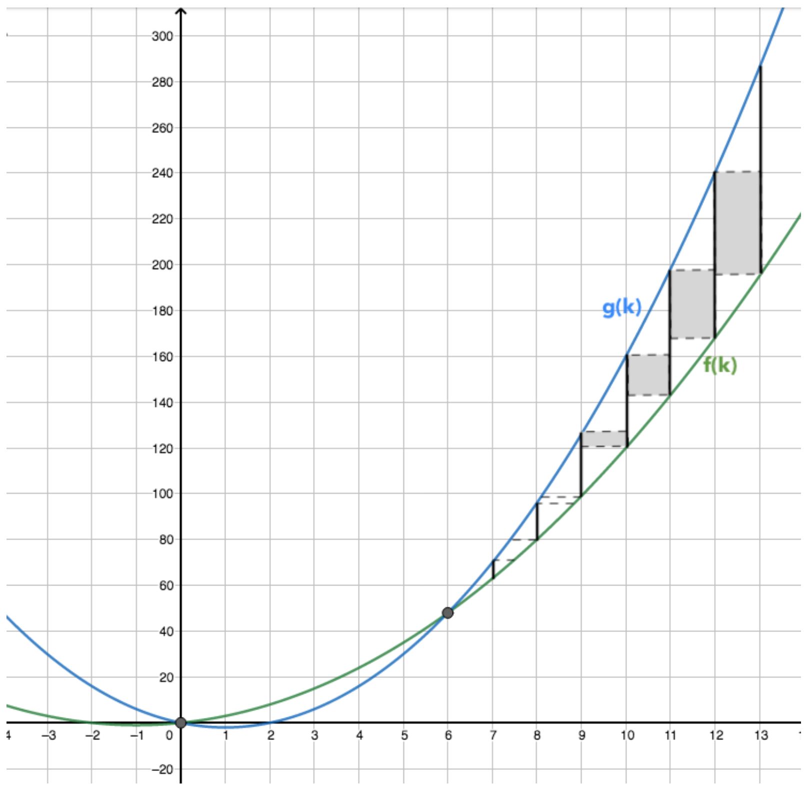
Frankly, this doesn't seem that hard to show. However, in the exam setting I could only come up with a not-so-straightforward way which I present in the solution below.

My solution to IMO P1: We first show that for some positive integer $k \geq 9$ the cards $(2k^2 - 4k), (2k^2 + 1)$ and $(2k^2 + 4k)$ exist in Ivan's deck. We then use the property that any two of these cards add up to a perfect square and the pigeonhole principle to prove the problem statement.

Lemma. *For every integer $n \geq 100$, there exists an integer $k \geq 9$ such that*

$$k^2 + 2k \leq n \leq 2k^2 - 4k$$

Proof. Let $f(x) = k^2 + 2k$ and $g(x) = 2k^2 - 4k$. We prove through a series of claims that the intervals $(f(k), g(k))$ and $(f(k+1), g(k+1))$ are overlapping, and that the interval sizes are non-decreasing. This is also visible in the graph, where the grey boxes denote the overlap in the intervals.



Claim 1. *$f(k+1) < g(k)$ for $k \geq 9$*

Proof. Consider another quadratic function $h(k) = k^2 - 8k - 3$. The coefficient of k^2 here is positive. Hence, $h(k)$ is an upward-facing quadratic. Additionally, by the quadratic formula, the roots of this quadratic occur before $k = 9$.

$$k = \frac{8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot (-3)}}{2} = 4 \pm \sqrt{19} \approx 8.36, -0.36 < 9$$

Thus, for all $k \geq 9$, $h(k) = k^2 - 8k - 3 > 0$

$$\begin{aligned}
&\implies 2k^2 - 4k > k^2 + 4k + 3 \\
&\implies 2k^2 - 4k > (k+1)^2 + 2(k+1) \\
&\implies g(k) > f(k+1) \\
&\implies f(k+1) < g(k)
\end{aligned}$$

□

Claim 2. *For $k \geq 9$, f and g are strictly increasing functions.*

Proof. Both f and g are upward-facing quadratic functions with their only turning points appearing before $k = 9$. The claim simply expresses this nature of the functions in different terms. □

It follows directly from Claim 1 and 2 that

$$f(k) < f(k+1) < g(k) < g(k+1)$$

for all $k \geq 9$. Thus, the intervals $(f(k), g(k))$ and $(f(k+1), g(k+1))$ are overlapping.

Claim 3. *For $k \geq 9$, the sizes of the interval between f and g increases.*

Proof. The function $g(k) - f(k)$ denotes the size of the intervals. It evaluates to $(2k^2 - 4k) - (k^2 + 2k) = k^2 - 6k$. This is an upward facing quadratic with vertex before $k = 9$, and hence a strictly increasing function of the domain considered. □

Thus, for any $n \geq 100$ there exists an interval of f and g where this n fits it. This is equivalent to $f(k) \leq n \leq g(k)$. ■

It follows from the lemma that for some k, $2k^2 - 4k \geq n$ and $2k^2 + 4k \leq 2n$. Since $n \leq 2k^2 - 4k < 2k^2 + 1 < 2k^2 + 4k \leq 2n$, the cards $(2k^2 - 4k)$, $(2k^2 + 1)$ and $(2k^2 + 4k)$ exist. But each pair from these three cards sum to a perfect square:

$$(2k^2 - 4k) + (2k^2 + 1) = (2k - 1)^2$$

$$(2k^2 - 4k) + (2k^2 + 4k) = (2k)^2$$

$$(2k^2 + 1) + (2k^2 + 4k) = (2k + 1)^2$$

Now, according to the pigeonhole principle, if these 3 cards are to be split into 2 piles, at least one pile would contain 2 of these cards. By selection, these cards add up to a perfect square. This proves the problem statement.

Remark. *If Claim 2 seems strange, even ingenious, you're not alone. Much like in the approach section, I worked this out backwards too. However, when writing proofs, it is good practice to write forwards only, since some logical deductions that worked well backwards may be dubious in the forward path. This ensures the proof remains rigorous, even if it hides the thought process behind seemingly "ingenious" steps.*

Remark. *We need the condition $n \geq 100$ because Claim 1 demands $k \geq 9$, which means the intervals may only begin from $f(9) = 9^2 + 2 \cdot 9 = 99$. In reality, the condition $n \geq 99$ works too, but it is less of a giveaway to say $n \geq 100$.*

3 Approaches to Preparation

Having understood the rigour of IMO preparation and the level of problem solving required, we are now in a good position to talk about the various approaches to preparation - broadly grouped into 3 categories.

Note on how to use this section

I personally think relying upon any generic approach (including those here) is recipe for disaster. By definition, these are not tailored to your strengths and weaknesses. Due to this, you would often end up learning from the wrong material in the wrong ways, at the wrong time. This is a huge waste of time, and part of the reason why I didn't learn as much as I could have in Strike 1.

Instead, I think it is much better to pick aspects and elements of these approaches that you think will work for you. This could be based on your existing knowledge, your problem solving abilities, the available time, or even the end goal (learning interesting math or winning competitions). You can then combine these elements up to create your own unique preparation strategy. It is up to you, how detailed you would like to make this. I initially started with a broad strategy in mind, and then as time progressed, started to frame it as weekly targets. You could do the same.

Pro tip: Ensure that you know why you're working through a certain resource in a certain way. This will help you stick to your strategy, thereby guaranteeing progress.

Let's now talk about the 3 approaches.

3.1 Casual

This approach is about going at a slow pace all round the year. You would spend between 25 minutes and an hour learning and doing math that interests you. For example, you could

- **Watch YouTube videos on math.** Some channels I watch are *Mind Your Decisions* (beginner), *Michael Penn* (intermediate), *Blackpenredpen* (intermediate), *Numberphile* and *3Blue1Brown*.
- **Do a Brilliant.org course.** This platform has a lot of fun, interactive, and bite-sized courses on all sorts of topics (including contest math). The platform is requires a small fee, but it is totally worth it.
- **Solve problems.** Here are some of the many sources:
  - Art of Problem Solving (AoPS) collection of contest problems (All range problems). This basically has problems from all contests across the world, including the AMC, the AIME, and the IMO. There are other problem collections available as books, but there really is no need to look beyond the AoPS collection.
  - CEMC Waterloo Past Contests
  - Australian Intermediate Math Contest (medium). Not many past contests are available online, but the problems are nice.
  - Mathematical Circles – A Russian Experience (easy-medium). This is a book (pdf available online) with a lot of interesting problems that will inspire discussions.

By doing these things, you will considerably develop your problem solving abilities, which is immensely important for both life and math competitions. Of course, if you wish to excel, you will also have to study in a focused manner.

This approach is also perfect if you don't wish to take competition math too seriously. This is because it is likely to be fun, and doesn't demand too much time. Moreover, doing these things improve your understanding of math as a whole, and enable you to appreciate that subject.

3.2 Focused but unstructured

In this approach, you would first start with a topic you want to learn about, and then find resources online to learn that material. Some useful resources are:

- **Wiki pages.** Wikipedia is a good option to understand a topic in detail. However, it can get too cryptic at times, and frequently contains details a beginner would not need to know. Brilliant.org wiki pages are more apt for such users - they're simpler and have problems incorporated throughout to keep you engaged. There are thousands of these. For example, here's the one on inequalities.
- Evan Chen's website. Evan Chen is a former IMO gold medalist who is now a Math Olympiad coach (USA).
- Yufei Zhao's handouts. This MIT professor has drafted numerous useful handouts for Math Olympiads.
- **YouTube videos.** If you find it hard to learn math by reading alone, this is your best bet. However, there's no one go to channel as far as I know. Just search your topic and look for a video.

To get you started, you can learn about the following topics. For further reference, see Evan Chen's informal syllabus of olympiad math.

- Modular Arithmetic
- Mathematical Induction
- Fermat's Little Theorem
- Angle Chase
- Pigeonhole Principle

This approach gets you used to reading math, and familiar with the core concepts. Since you will be learning things when you encounter them, you can be sure that what you're learning is actually useful for contests. The disadvantage is that your knowledge will not be very comprehensive and it may seem a bit haphazard sometimes.

This approach is quite common. In fact, I often try to understand difficult material (such as research papers) by applying this approach to the key terms I don't understand. The approach also worked out well for many of my peers. However, as seen in Strike 1, in the long run, a comprehensive understanding becomes important. This leads us to the approach 3.

3.3 Focused and structured

This approach involves adopting a rigorous curriculum which you will work on. There are two options here:

1. Enroll in a course designed for this purpose. A popular option is the AoPS WOOT programme. You could also find some teachers and take classes from them. However, this option usually comes with a high price tag, and like me, many people simply cannot afford it. A good and affordable (but slow-paced) option is the brilliant.org contest math course.
2. Develop your own curriculum using freely available books. Apart from being free, this is self-paced and tailored to your own needs (you design it!).

If you choose the second option, it is really important to have a good selection of books, and a good study plan. The internet is flooding with books on Olympiad Math; I myself found over 50. Furthermore, every other person will recommend a different selection of books, each of which seem better than what you're currently using. Amidst all this, do not be compelled to change your books. This is a huge waste of time and effort. Instead, fix one selection of books and stick to it until your targets have been met. To ensure you make the right selections, you should go through each of the books and see how they are structured and which ones feel better to you. Once you've selected your books, identify what chapters you wish to do, and set reasonable weekly targets.

Here are some book recommendations from my side (free PDFs are generally available on the internet). I do think these are the best options out there, but you should make your own choices.

- Modern Olympiad Number Theory (MONT) - only first 4 chapters needed.
- Euclidean Geometry in Mathematical Olympiads (EGMO) - only first 3/4 chapters needed.
- Inequalities: Go through Evan Chen's handout, followed by Thomas Mildorf's handout - do as much as possible.

Combinatorics and Functional Equations do not require much theoretical knowledge – they're more about problem solving skills. You could develop these by doing a lot of problems, and discovering the tools and approaches for yourself. Alternatively, you could resort to books, which organise these problem solving strategies. The suggestions below don't clearly stand out as *the best*, but they're good nonetheless:

- Olympiad Combinatorics by Pranav Sriram
- Functional Equations - a problem solving approach by B J Venkatachala
- Functional Equations and How to Solve Them by Christopher G. Small

Whatever books you choose, ensure you do a considerable amount of the end-of-chapter problems, even if you understand all the theory perfectly. This is because problems indicate how the theory applies to different situations. Once done with a decent amount of theory (and topic-specific problems) in any given field (2-3 chapters), you should begin attempting previous contest problems (such as the IMO shortlists) in parallel to learning theory. As you saw in section 2, hard problems typically require multiple stages of insights. Without practice, you will be clueless when faced with them.

Key Takeaways

I've said a lot in the previous pages, and so, I'd like to reinforce some of the key takeaways. Simple as they sound, these are exactly the things one has trouble with during preparation.

1. **It's a significant time investment.** Make sure you're mentally prepared to invest approximately 500 hours into this. That's my estimate of how long it will take to become as good as an IMO gold medalist.
2. **You may not make it to the IMO, but the journey will still be worth it.** How you perform on the final tests depends as much on luck as on skill. Thus, there's a chance you may not make it. Nonetheless, you will still emerge as a much better problem solver, you will have gained a new perspective to math, and most importantly, you would've enjoyed it.
3. **Do what interests you.** More than prestige, math olympiads are about indulging in interesting mathematics and having fun. If you like solving problems more than learning theory, do so! If you like working collaboratively more than studying alone, do so! Unsurprisingly, your preparation will be more effective if you enjoy it.
4. **Your preparation is *entirely* in your hands.** You should not be overly-reliant or overly-expectant from the training camps. Create a good preparation strategy for yourself, and stick to it. Training camps should only form a part of your preparation strategy.
5. **Don't hesitate to reach out for help**, especially if you need suggestions on how to prepare. As explained here, ensure you explain your situation well to get useful results.

So. . . what are you waiting for? Get started!