Digression on type safety
18.02 Recitation (MW9)

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In mathematics, statements are usually either true or false. Examples of false statements[1] include
$$\pi = \frac{16}{5} \qquad \text{or} \qquad 2 + 2 = 5.$$

However, it's possible to write statements that are not merely false, but not even "grammatically correct", such as the nonsense equations
$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad |\mathbf{i} + 3\mathbf{j}| = \cos(\mathbf{k}), \qquad \det \begin{pmatrix} 5 \\ 11 \end{pmatrix} \neq \sqrt{2}.$$

To call these equations false is misleading. If your friend asked you whether $2 + 2 = 5$, you would say "no". But if your friend asked whether π equals the 2×2 identity matrix, the answer is a different kind of "no"; really, it's "your question makes no sense".

These three examples are **type errors**. This term comes from programming: most programming languages have different data types like integer, boolean, string, array, etc., and will usually[2] prevent you from doing anything idiotic like adding a string to an array.

Objects in mathematics work in a really similar way. In the first weeks of 18.02, you will meet real numbers, vectors, and matrices; these are all different types of objects, and certain operations are **defined** (aka "allowed") or **undefined** (aka "not allowed") depending on the underlying types. Table 1 lists some common examples.

Operation	Notation	Input 1	Input 2	Output
Add/subtract	$a + b$	Scalar	Scalar	Scalar
Add/subtract	$\vec{v} + \vec{w}$	Length d vector	Length d vector	Length d vector
Add/subtract	$M + N$	$m \times n$ matrix	$m \times n$ matrix	$m \times n$ matrix
Multiplication	$c\vec{v}$	Scalar	Length d vector	Length d vector
Multiplication	ab	Scalar	Scalar	Scalar
Multiplication	MN	$m \times n$ matrix	$n \times p$ matrix	$m \times p$ matrix
Dot product	$\vec{v} \cdot \vec{w}$	Length d vector	Length d vector	Scalar
Cross product	$\vec{v} \times \vec{w}$	Length 3 vector	Length 3 vector	Length 3 vector
Length/mag.	$|\vec{v}|$	Any vector		Scalar
Determinant	$\det A$	Any square matrix		Scalar

Table 1: Common linalg operations. For 18.02, "scalar" and "real number" are synonyms.

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[1] Indiana Pi bill and 1984, respectively.

[2] JavaScript is a notable exception. In JavaScript, you may know that `[]` and `{}` are an empty array and an empty object, respectively. Then `[]+[]` is the empty string, `[]+{}` is the string `` `[object Object]' ``, `{}+[]` is `0`, and `{}+{}` is `NaN` (not a number).

A common question at this point is how you are supposed to figure out whether a certain operation is allowed or not. For example, many students want to try and multiply two vectors together component-wise; why is

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

not a legal sentence? It seems like it would make sense.

The answer is that you *don't* have to figure out — you are *told*. Table 1 isn't something that you derive. Instead, it's the set of *definitions* which you have been given. Or more sarcastically, it's all just a social construct.[3]

Why you should care

What this means is that, every time you encounter a new kind of mathematical object or operation (e.g. partial derivative), one of the first things that you might want to do is figure out what kind of types it is permitted to interact with. This helps give you a sanity check on your understanding of the new concept.

Practically, what's really useful is that if you have a good handle on types, then it gives you way to do **type checking** on your work. This is the analog of dimensional analysis from 8.01/8.02, where you know you messed up if some equation has $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$ on the left but $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$ on the right.

For example, if you are checking your work and you see something like

$$|\vec{v} \times \vec{p}| = 9\vec{p}$$

then you can immediately tell that there's a mistake, because the two sides are incompatible — the left-hand side is a real number, but the right-hand side is a vector.

References

- *The Type System of Mathematics*, Qiaochu Yuan, https://qchu.wordpress.com/2013/05/28/the-type-system-of-mathematics/.

- *Basic Category Theory*, Tom Leinster, Chapter 3. https://arxiv.org/abs/1612.09375

Hyperlink

You can find a copy of this file at https://web.evanchen.cc/1802.html. I'll also upload the slides for today there.

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[3]Of course, it's not quite as arbitrary as it sounds. The long explanation goes something like this. Definitions aren't judged by "correctness", because that doesn't even make sense; you are allowed to make up whatever definitions you want. Instead, definitions are judged by whether they are *useful*. Which is obviously subjective, but it's less subjective than you might guess.