Quick answer key to Recitation 23

ChatGPT 4o

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Computing the Curl of a Vector Field

We are tasked with finding the curl of the vector field:

$$\mathbf{V}(x,y,z) = \langle x^2y,\ yz,\ xyz^2 \rangle$$

1.1 Definition of Curl

The curl of a vector field $\mathbf{V} = \langle P, Q, R \rangle$ is given by:

$$\nabla \times \mathbf{V} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Alternatively, using the determinant form:

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

1.2 Components of the Vector Field

Given:

$$P = x^2 y, \quad Q = yz, \quad R = xyz^2$$

1.3 Calculating the Partial Derivatives

Component i:

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(xyz^2) = xz^2$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(yz) = y$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = xz^2 - y$$

Component j:

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(xyz^2) = yz^2$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(x^2y) = 0$$

$$\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = yz^2 - 0 = yz^2$$

$$-\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = -yz^2$$

Component k:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2y) = x^2$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 - x^2 = -x^2$$

1.4 Assembling the Curl

Combining the components, we obtain:

$$\nabla \times \mathbf{V} = \langle xz^2 - y,\ -yz^2,\ -x^2 \rangle$$

1.5 Final Answer

The curl of the vector field $\mathbf{V}(x, y, z) = \langle x^2y,\ yz,\ xyz^2 \rangle$ is:

$$\boxed{\nabla \times \mathbf{V} = \langle xz^2 - y,\ -yz^2,\ -x^2 \rangle}$$

2 Conservative Vector Fields and Line Integrals

We are given the vector field:

$$\mathbf{F}(x, y, z) = \langle yz^2,\ xz^2 + ayz,\ bxyz + y^2 \rangle$$

and a parametrized curve:

$$\mathbf{C}(t) = \left(e^{t^2 - t} - 1,\ t^4,\ \sin(\pi t)\right), \quad 0 \leq t \leq 1$$

We aim to determine the values of a and b for which $\mathbf{F}$ is a conservative vector field, find a corresponding potential function $f(x, y, z)$, and compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the fundamental theorem of calculus for line integrals.

2.1 Part (a): Determining Values of a and b for Conservativity

A vector field $\mathbf{F}$ is conservative if there exists a scalar potential function f such that $\mathbf{F} = \nabla f$. A necessary condition for conservativity in simply connected domains is that the curl of $\mathbf{F}$ is zero everywhere:

$$\nabla \times \mathbf{F} = \mathbf{0}$$

Computing the Curl of F

Given:

$$\mathbf{F} = \langle yz^2,\ xz^2 + ayz,\ bxyz + y^2 \rangle$$

The curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Where:

$$P = yz^2, \quad Q = xz^2 + ayz, \quad R = bxyz + y^2$$

Calculating each component:

Component i:

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(bxyz + y^2) = bxz + 2y$$

$$\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(xz^2 + ayz) = 2xz + ay$$

$$\text{Curl}_x = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = (bxz + 2y) - (2xz + ay) = (b-2)xz + (2-a)y$$

Component j:

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(bxyz + y^2) = byz$$

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}(yz^2) = 2yz$$

$$\text{Curl}_y = -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = -(byz - 2yz) = -(b-2)yz$$

Component k:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xz^2 + ayz) = z^2$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(yz^2) = z^2$$

$$\text{Curl}_z = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = z^2 - z^2 = 0$$

Thus, the curl of $\mathbf{F}$ is:

$$\nabla \times \mathbf{F} = [(b-2)xz + (2-a)y]\,\mathbf{i} - (b-2)yz\mathbf{j} + 0\mathbf{k}$$

Setting the Curl to Zero

For $\mathbf{F}$ to be conservative, $\nabla \times \mathbf{F} = \mathbf{0}$ for all x, y, z. Therefore:

$$(b-2)xz + (2-a)y = 0 \quad \text{and} \quad -(b-2)yz = 0$$

for all x, y, z.

From the second equation:

$$-(b-2)yz = 0$$

Since this must hold for all y and z, we must have:

$$b - 2 = 0 \quad \Rightarrow \quad b = 2$$

Substituting $b = 2$ into the first equation:

$$(2-2)xz + (2-a)y = 0 \quad \Rightarrow \quad (2-a)y = 0$$

For this to hold for all y, we must have:

$$2 - a = 0 \quad \Rightarrow \quad a = 2$$

2.2 Part (a) Answer

The vector field $\mathbf{F} = \langle yz^2,\ xz^2 + ayz,\ bxyz + y^2 \rangle$ is conservative if and only if:

$$\boxed{a = 2 \quad \text{and} \quad b = 2}$$

2.3 Part (b): Finding the Potential Function $f(x, y, z)$

Given $a = 2$ and $b = 2$, the vector field becomes:

$$\mathbf{F} = \langle yz^2,\ xz^2 + 2yz,\ 2xyz + y^2 \rangle$$

We seek a scalar function $f(x, y, z)$ such that:

$$\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

Integrating $\frac{\partial f}{\partial x} = yz^2$ with respect to x

$$f(x, y, z) = \int yz^2 \, dx + g(y, z) = yz^2 x + g(y, z)$$

Determining $g(y, z)$ by Differentiating with Respect to y

$$\frac{\partial f}{\partial y} = z^2 x + \frac{\partial g}{\partial y} = xz^2 + 2yz$$
$$\Rightarrow \frac{\partial g}{\partial y} = 2yz$$
$$\Rightarrow g(y, z) = \int 2yz \, dy + h(z) = y^2 z + h(z)$$

Updating $f(x, y, z)$

$$f(x, y, z) = yz^2 x + y^2 z + h(z)$$

Determining $h(z)$ by Differentiating with Respect to z

$$\frac{\partial f}{\partial z} = 2yzx + y^2 + \frac{dh}{dz} = 2xyz + y^2$$
$$\Rightarrow 2xyz + y^2 + \frac{dh}{dz} = 2xyz + y^2$$
$$\Rightarrow \frac{dh}{dz} = 0$$
$$\Rightarrow h(z) = \text{constant} = C$$

For simplicity, set $C = 0$.

Final Potential Function

$$\boxed{f(x, y, z) = xyz^2 + y^2 z}$$

2.4 Part (c): Calculating the Line Integral Using the Potential Function

Given the parametrized curve:

$$\mathbf{C}(t) = \left(e^{t^2-t} - 1,\ t^4,\ \sin(\pi t)\right), \quad 0 \leq t \leq 1$$

Since $\mathbf{F}$ is conservative and $\mathbf{F} = \nabla f$, the fundamental theorem for line integrals states:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{C}(1)) - f(\mathbf{C}(0))$$

Determining the Endpoints

$$\mathbf{C}(0) = \left(e^{0-0} - 1,\ 0^4,\ \sin(0)\right) = (1-1,\ 0,\ 0) = (0,0,0)$$
$$\mathbf{C}(1) = \left(e^{1-1} - 1,\ 1^4,\ \sin(\pi \cdot 1)\right) = (1-1,\ 1,\ 0) = (0,1,0)$$

Evaluating the Potential Function at the Endpoints

$$f(0,0,0) = 0 \cdot 0 \cdot 0^2 + 0^2 \cdot 0 = 0 + 0 = 0$$
$$f(0,1,0) = 0 \cdot 1 \cdot 0^2 + 1^2 \cdot 0 = 0 + 0 = 0$$

Calculating the Line Integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,1,0) - f(0,0,0) = 0 - 0 = 0$$

2.5 Final Answer

(a) The vector field $\mathbf{F} = \langle yz^2,\ xz^2 + ayz,\ bxyz + y^2 \rangle$ is conservative if and only if:

$$\boxed{a = 2 \quad \text{and} \quad b = 2}$$

(b) For $a = 2$ and $b = 2$, a corresponding potential function is:

$$\boxed{f(x, y, z) = xyz^2 + y^2 z}$$

(c) The line integral of $\mathbf{F}$ along the curve C is:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0}$$

3 Verification of Stokes' Theorem

We aim to verify Stokes' Theorem for the following scenario:

- **Surface S:** The upper hemisphere of the unit sphere centered at the origin, defined by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.
- **Curve C:** The boundary of S, which is the unit circle in the xy-plane, defined by $x^2 + y^2 = 1$ and $z = 0$.
- **Vector Field $\mathbf{F}$:** $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.

Stokes' Theorem states that:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where:

- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$.
- $\mathbf{n}$ is the unit normal vector to the surface S.
- dS is the differential element of the surface area.
- $d\mathbf{r}$ is the differential element of the curve C.

Our goal is to compute both the surface integral of the curl of $\mathbf{F}$ over S and the line integral of $\mathbf{F}$ around C, and verify that they are equal.

3.1 Step 1: Compute the Curl of F

Given the vector field:

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle$$

The curl of $\mathbf{F}$ is computed as:

$$\nabla \times \mathbf{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

Substituting the components of $\mathbf{F}$:

$$\frac{\partial F_z}{\partial y} = \frac{\partial z}{\partial y} = 0, \quad \frac{\partial F_y}{\partial z} = \frac{\partial y}{\partial z} = 0$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial x}{\partial z} = 0, \quad \frac{\partial F_z}{\partial x} = \frac{\partial z}{\partial x} = 0$$
$$\frac{\partial F_y}{\partial x} = \frac{\partial y}{\partial x} = 0, \quad \frac{\partial F_x}{\partial y} = \frac{\partial x}{\partial y} = 0$$

Thus, the curl simplifies to:

$$\nabla \times \mathbf{F} = \langle 0 - 0,\ 0 - 0,\ 0 - 0 \rangle = \langle 0, 0, 0 \rangle$$

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}$$

3.2 Step 2: Compute the Surface Integral of the Curl

Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the surface integral becomes:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{0} \cdot \mathbf{n}\, dS = 0$$

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0}$$

3.3 Step 3: Compute the Line Integral

To compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, we parametrize the curve C.

Parametrization of C: Since C is the unit circle in the xy-plane, we can parametrize it using the parameter t as follows:

$$\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle, \quad 0 \leq t \leq 2\pi$$

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$$

Evaluating $\mathbf{F}$ Along C:

$$\mathbf{F}(\mathbf{r}(t)) = \langle \cos t, \sin t, 0 \rangle$$

Dot Product $\mathbf{F} \cdot \mathbf{r}'(t)$:

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle \cos t, \sin t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = -\cos t \sin t + \sin t \cos t + 0 = 0$$

Line Integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 0 \, dt = 0$$

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = 0}$$

3.4 Conclusion

Both the surface integral of the curl of $\mathbf{F}$ over S and the line integral of $\mathbf{F}$ around C are zero. Therefore, Stokes' Theorem holds for this vector field and surface.

$$\boxed{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = 0}$$