

# Notes for 18.02 Recitation 23

## 18.02 Recitation MW9

EVAN CHEN

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*My point today is that, if we wish to count lines of code, we should not regard them as “lines produced” but as “lines spent”.*

— Edsger W. Dijkstra, EWD 1036

This handout (and any other DLC’s I write) are posted at <https://web.evanchen.cc/1802.html>.

- Course evaluations are available at <https://registrar.mit.edu/subjectevaluation> and will be open until Monday, December 16 at 9AM.
- Tue/Wed will be review for final; Wed is the last day of class. Practice problems out now.

### S1 Curl

This is the final red arrow in the poster <https://web.evanchen.cc/textbooks/poster-stokes.pdf>.

#### Definition of curl

Suppose  $\mathbf{F}(x, y, z) = \langle p, q, r \rangle$  is a 3D vector field. Then the **curl** of  $\mathbf{F}$  is the vector field defined by

$$\text{curl } \mathbf{F} := \nabla \times \mathbf{F} := \begin{pmatrix} \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z} \\ \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \end{pmatrix}.$$

Type signature: The curl takes in only a 3D vector field. The curl at each point is a 3D vector (i.e. the curl of a 3D vector field is itself a 3D vector field).

#### Tip: How to memorize curl

In practice, everyone remembers this formula using the following mnemonic:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix}.$$

(This equation does not pass type-safety, because it’s a “matrix” whose entries are some combination of functions, vectors, and partial derivative operators, so it absolutely does not make sense.) This is why  $\nabla \times \mathbf{F}$  is the notation chosen; you could almost imagine  $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$  in which case the determinant above is the old mnemonic for the cross product.

## §2 Line integrals in 3D are pretty much the same as 2D

- Definition of the line integral is unchanged from 2D; it is still  $\int_{t=\text{start time}}^{\text{stop time}} \mathbf{F} \cdot \mathbf{r}'(t) dt$ .
- A vector field on all of  $\mathbb{R}^3$  is *conservative* (i.e.  $\mathbf{F} = \nabla f$  for some potential function  $f$ ) if and only if  $\nabla \times \mathbf{F} = \mathbf{0}$ .
  - This time there are three equations, one for each component:

$$\frac{\partial r}{\partial y} = \frac{\partial q}{\partial z}, \quad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x}, \quad \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

The above three equations should be remembered as  $f_{xy} = f_{yx}$ ,  $f_{yz} = f_{zy}$ ,  $f_{zy} = f_{xy}$ .

- In contrast, for the 2D case, there was only one equation  $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ . The contrast is because 2D scalar curl was a number but curl is a vector in  $\mathbb{R}^3$ .
- Assuming the vector field is conservative, the anti-gradient procedure for potential function is the same as before. See Chapter 16 of my LAMV notes.
- Once you have a potential function, FTC still works fine.

## §3 Crummy Stokes' theorem (not on final)



### Definition of compatible orientations

Suppose  $\mathcal{C}$  is a closed loop in  $\mathbb{R}^3$  which is the boundary of an oriented surface  $\mathcal{S}$ . The orientation of  $\mathcal{C}$  and  $\mathcal{S}$  are *compatible* if, when walking along  $\mathcal{C}$  in the chosen direction, with  $\mathcal{S}$  to the left, the normal vector  $\mathbf{n}$  is pointing up.

**Theorem** (Crummy Stokes' theorem). *Let  $\mathcal{C}$  be a closed loop in  $\mathbb{R}^3$  parametrized by  $\mathbf{r}_1(t)$ . Suppose  $\mathcal{S}$  is the boundary of an oriented surface  $\mathcal{S}$  parametrized by  $\mathbf{r}_2(u, v)$ . Assume the orientation of  $\mathcal{C}$  and  $\mathcal{S}$  are compatible. Then*

$$\underbrace{\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}_1}_{= \int_{t=\text{start}}^{\text{stop}} \mathbf{F} \cdot \mathbf{r}'_1(t) dt} = \underbrace{\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS}_{= \iint_{u,v} (\text{curl } \mathbf{F}) \cdot \left( \frac{\partial \mathbf{r}_2}{\partial u} \times \frac{\partial \mathbf{r}_2}{\partial v} \right) du dv}$$

Reasons to not be excited about the non-generalized Stokes' theorem (hence the name “crummy”):

- Both sides require parametrization, so it's not as slick as FTC, Green, or divergence theorem, which were powerful because they let you skip the parametrization step.
- Surface integrals are more painful than line integrals, but there's no “anti-curl” procedure analogous to anti-gradient, so it doesn't help with surface integrals of a “random” vector field.

## §4 Recitation questions from official course

1. Find the curl of the vector field  $\langle x^2y, yz, xyz^2 \rangle$ .
2. For what values of  $a$  and  $b$  is the vector field  $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + ayz)\mathbf{j} + (bxyz + y^2)\mathbf{k}$  a conservative field? For these values, find a corresponding potential function  $f(x, y, z)$  for the vector field. Calculate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $C$  is the parametrized curve  $(e^{t^2-t} - 1, t^4, \sin(\pi t))$  with  $0 \leq t \leq 1$ .
3. Recall the statement of Stokes' theorem. Verify the statement of Stokes' theorem where  $S$  is the upper hemisphere of the sphere of radius one centered at the origin and  $C$  is its boundary, for the vector field  $\mathbf{F} = \langle x, y, z \rangle$ .