Notes for 18.02 Recitation 23

18.02 Recitation MW9

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My point today is that, if we wish to count lines of code, we should not regard them as "lines produced" but as "lines spent".

— *Edsger W. Dijkstra, EWD 1036*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

- Course evaluations are available at https://registrar.mit.edu/subjectevaluation and will be open until Monday, December 16 at 9AM.
- Tue/Wed will be review for final; Wed is the last day of class. Practice problems out now.

§1 Curl

This is the final red arrow in the poster https://web.evanchen.cc/textbooks/poster-stokes.pdf.

Definition of curl

Suppose $\mathbf{F}(x, y, z) = \langle p, q, r \rangle$ is a 3D vector field. Then the **curl** of $\mathbf{F}$ is the vector field defined by

$$\operatorname{curl} \mathbf{F} := \nabla \times \mathbf{F} := \begin{pmatrix} \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z} \\ \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \end{pmatrix}.$$

Type signature: The curl takes in only a 3D vector field. The curl at each point is a 3D vector (i.e. the curl of a 3D vector field is itself a 3D vector field).

Tip: How to memorize curl

In practice, everyone remembers this formula using the following mnemonic:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix}.$$

(This equation does not pass type-safety, because it's a "matrix" whose entries are some combination of functions, vectors, and partial derivative operators, so it absolutely does not make sense.) This is why $\nabla \times \mathbf{F}$ is the notation chosen; you could almost imagine $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ in which case the determinant above is the old mnemonic for the cross product.

§2 Line integrals in 3D are pretty much the same as 2D

- Definition of the line integral is unchanged from 2D; it is still $\int_{t=\text{start time}}^{\text{stop time}} \mathbf{F} \cdot \mathbf{r}'(t)\,\mathrm{d}t$.
- A vector field on all of $\mathbb{R}^3$ is *conservative* (i.e. $\mathbf{F} = \nabla f$ for some potential function f) if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.
  - This time there are three equations, one for each component:

$$\frac{\partial r}{\partial y} = \frac{\partial q}{\partial z}, \quad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x}, \quad \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

    The above three equations should be remembered as $f_{xy} = f_{yx}$, $f_{yz} = f_{zy}$, $f_{zy} = f_{xy}$.
  - In contrast, for the 2D case, there was only one equation $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$. The contrast is because 2D scalar curl was a number but curl is a vector in $\mathbb{R}^3$.
- Assuming the vector field is conservative, the anti-gradient procedure for potential function is the same as before. See Chapter 16 of my LAMV notes.
- Once you have a potential function, FTC still works fine.

§3 Crummy Stokes' theorem (not on final)

Definition of compatible orientations

Suppose $\mathcal{C}$ is a closed loop in $\mathbb{R}^3$ which is the boundary of an oriented surface $\mathcal{S}$. The orientation of $\mathcal{C}$ and $\mathcal{S}$ are *compatible* if, when walking along $\mathcal{C}$ in the chosen direction, with $\mathcal{S}$ to the left, the normal vector $\mathbf{n}$ is pointing up.

Theorem (Crummy Stokes' theorem)**.** *Let $\mathcal{C}$ be a closed loop in $\mathbb{R}^3$ parametrized by $\mathbf{r}_1(t)$. Suppose $\mathcal{S}$ is the boundary of an oriented surface $\mathcal{S}$ parametrized by $\mathbf{r}_2(u,v)$. Assume the orientation of $\mathcal{C}$ and $\mathcal{S}$ are compatible. Then*

$$\underbrace{\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}_1}_{= \int_{t=\text{start}}^{\text{stop}} \mathbf{F} \cdot \mathbf{r}_1'(t)\,\mathrm{d}t} = \underbrace{\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S.}_{= \iint_{u,v} (\operatorname{curl} \mathbf{F}) \cdot \left(\frac{\partial \mathbf{r}_2}{\partial u} \times \frac{\partial \mathbf{r}_2}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v}$$

Reasons to not be excited about the non-generalized Stokes theorem (hence the name "crummy"):

- Both sides require parametrization, so it's not as slick as FTC, Green, or divergence theorem, which were powerful because they let you skip the parametrization step.
- Surface integrals are more painful than line integrals, but there's no "anti-curl" procedure analogous to anti-gradient, so it doesn't help with surface integrals of a "random" vector field.

§4 Recitation questions from official course

1. Find the curl of the vector field $\langle x^2y, yz, xyz^2 \rangle$.
2. For what values of a and b is the vector field $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + ayz)\mathbf{j} + (bxyz + y^2)\mathbf{k}$ a conservative field? For these values, find a corresponding potential function $f(x,y,z)$ for the vector field. Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the parametrized curve $\left(e^{t^2-t} - 1, t^4, \sin(\pi t)\right)$ with $0 \le t \le 1$.
3. Recall the statement of Stokes' theorem. Verify the statement of Stokes theorem where S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary, for the vector field $\mathbf{F} = \langle x, y, z \rangle$.