Quick answer key to Recitation 21

ChatGPT 4o

2 December 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Flux of the Vector Field Through a Conical Surface

We are tasked with computing the upward flux of the vector field $\mathbf{V}(x, y, z) = \langle x, y, 2z \rangle$ through the surface S, which is the portion of the cone $z = \sqrt{x^2 + y^2}$ lying in the region $1 \leq z \leq 2$.

1.1 Understanding the Surface S

- **Equation of the Cone:** $z = \sqrt{x^2 + y^2}$ or equivalently $z^2 = x^2 + y^2$.
- **Bounds:** The region of interest lies between $z = 1$ and $z = 2$.
- **Intersection with Planes:** At $z = 1$, the cone intersects the plane $z = 1$ at $x^2 + y^2 = 1$. At $z = 2$, it intersects at $x^2 + y^2 = 4$.

1.2 Parametrization of the Surface S

We will use cylindrical coordinates to parametrize the surface S.

Cylindrical Coordinates Overview

In cylindrical coordinates (r, θ, z):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where:

- $r \geq 0$ is the radial distance,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- z is the height.

Parametrization

Given the cone $z = r$, the surface S can be parametrized as:

$$\mathbf{r}(r, \theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle$$

with the bounds:

$$1 \leq r \leq 2, \quad 0 \leq \theta < 2\pi$$

1.3 Computing the Tangent Vectors

To find the flux, we need the upward-pointing normal vector. We obtain this by computing the cross product of the tangent vectors.

Partial Derivative with respect to r:

$$\mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r} = \langle \cos\theta,\, \sin\theta,\, 1 \rangle$$

Partial Derivative with respect to θ:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r\sin\theta,\, r\cos\theta,\, 0 \rangle$$

Cross Product $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta,\, -r\sin\theta,\, r \rangle$$

This vector points upward as the z-component is positive.

1.4 Calculating the Flux

The flux Φ of $\mathbf{V}$ through S is given by:

$$\Phi = \iint_S \mathbf{V} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}\, dS = \mathbf{r}_r \times \mathbf{r}_\theta\, dr\, d\theta = \langle -r\cos\theta,\, -r\sin\theta,\, r\rangle\, dr\, d\theta$.

Dot Product $\mathbf{V} \cdot \mathbf{n}$

Evaluate $\mathbf{V}$ at $\mathbf{r}(r, \theta)$:

$$\mathbf{V}(x, y, z) = \langle x, y, 2z\rangle = \langle r\cos\theta,\, r\sin\theta,\, 2r\rangle$$

Dot product with $\mathbf{n}$:

$$\mathbf{V} \cdot \mathbf{n} = \langle r\cos\theta,\, r\sin\theta,\, 2r\rangle \cdot \langle -r\cos\theta,\, -r\sin\theta,\, r\rangle$$

$$= -r^2\cos^2\theta - r^2\sin^2\theta + 2r^2 = r^2(-\cos^2\theta - \sin^2\theta + 2)$$

$$= r^2(-1 + 2) = r^2$$

Thus, the integrand simplifies to r^2.

Setting Up the Integral

The flux integral becomes:

$$\Phi = \int_0^{2\pi} \int_1^2 r^2\, dr\, d\theta$$

Evaluating the Integral

Integrate with respect to r:

$$\int_1^2 r^2\, dr = \left[\frac{r^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Integrate with respect to θ:

$$\int_0^{2\pi} \frac{7}{3}\, d\theta = \frac{7}{3} \times 2\pi = \frac{14\pi}{3}$$

1.5 Final Answer

The upward flux of $\mathbf{V} = \langle x, y, 2z \rangle$ through the surface S is:

$$\boxed{\Phi = \frac{14\pi}{3}}$$

2 Link to second question

Supplementary material V9, example 1, on page 2.

https://ocw.mit.edu/courses/18-02-multivariable-calculus-fall-2007/b99f7b21fccbe519c0839995574b4da8_surface_integrls.pdf

3 Link to third question

Supplementary material V9, example 2, on page 3.

https://ocw.mit.edu/courses/18-02-multivariable-calculus-fall-2007/b99f7b21fccbe519c0839995574b4da8_surface_integrls.pdf

4 Flux of the Vector Field Through a Cylindrical Surface

We are tasked with computing the outward flux of the vector field

$$\mathbf{V}(x, y, z) = \langle x^3z^2 + y^2z,\ x^2yz^2 - xyz,\ xz^4 - y^5\rangle$$

through the surface S, which is the portion of the cylinder $x^2 + y^2 = 1$ in the first octant ($x, y, z \geq 0$) that lies below $z = 1$.

4.1 Understanding the Surface S

- **Equation of the Cylinder:** $x^2 + y^2 = 1$
- **Bounds:**
$$0 \leq z \leq 1$$
- **Octant Restriction:** $x, y, z \geq 0$ implies:
$$0 \leq \theta \leq \frac{\pi}{2} \quad \text{(in cylindrical coordinates)}$$

4.2 Parametrization of the Surface S

We will use cylindrical coordinates (r, θ, z) to parametrize the surface S.

Cylindrical Coordinates Overview

In cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where:

- $r \geq 0$ is the radial distance,
- $0 \leq \theta \leq \frac{\pi}{2}$ (first octant),
- $0 \leq z \leq 1$.

Parametrization

Since the cylinder $x^2+y^2 = 1$ corresponds to $r = 1$ in cylindrical coordinates, the parametrization of S is:

$$\mathbf{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z \rangle$$

with:

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq z \leq 1$$

4.3 Computing the Tangent Vectors

To compute the flux, we need the outward-pointing unit normal vector. We obtain this by computing the cross product of the tangent vectors.

Partial Derivative with respect to θ:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -\sin\theta,\ \cos\theta,\ 0\rangle$$

Partial Derivative with respect to z:

$$\mathbf{r}_z = \frac{\partial \mathbf{r}}{\partial z} = \langle 0,\ 0,\ 1\rangle$$

Cross Product $\mathbf{r}_\theta \times \mathbf{r}_z$:

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}(\cos\theta \cdot 1 - 0 \cdot 0) - \mathbf{j}(-\sin\theta \cdot 1 - 0 \cdot 0) + \mathbf{k}(-\sin\theta \cdot 0 - \cos\theta \cdot 0)$$

$$= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + 0\,\mathbf{k} = \langle \cos\theta,\ \sin\theta,\ 0\rangle$$

This vector points outward from the cylinder.

4.4 Calculating the Flux

The outward flux Φ of $\mathbf{V}$ through S is given by:

$$\Phi = \iint_S \mathbf{V} \cdot \mathbf{n} \, dS$$

where $\mathbf{n}\, dS = \mathbf{r}_\theta \times \mathbf{r}_z \, d\theta \, dz = \langle \cos\theta, \ \sin\theta, \ 0 \rangle \, d\theta \, dz$.

Expressing V in Cylindrical Coordinates

At a point on S, $x = \cos\theta$, $y = \sin\theta$, and $z = z$. Thus, the vector field $\mathbf{V}$ becomes:

$$\mathbf{V}(x, y, z) = \langle (\cos\theta)^3 z^2 + (\sin\theta)^2 z, \ (\cos\theta)^2 \sin\theta z^2 - \cos\theta \sin\theta z, \ \cos\theta z^4 - (\sin\theta)^5 \rangle$$

Dot Product V · n

Compute $\mathbf{V} \cdot \mathbf{n}$:

$$\mathbf{V} \cdot \mathbf{n} = \langle (\cos\theta)^3 z^2 + (\sin\theta)^2 z, \ (\cos\theta)^2 \sin\theta z^2 - \cos\theta \sin\theta z, \ \cos\theta z^4 - (\sin\theta)^5 \rangle \cdot \langle \cos\theta, \ \sin\theta, \ 0 \rangle$$

$$= \big((\cos\theta)^3 z^2 + (\sin\theta)^2 z\big) \cos\theta + \big((\cos\theta)^2 \sin\theta z^2 - \cos\theta \sin\theta z\big) \sin\theta + \big(\cos\theta z^4 - (\sin\theta)^5\big) \cdot 0$$

$$= (\cos^4\theta z^2 + \cos\theta \sin^2\theta z) + (\cos^2\theta \sin^2\theta z^2 - \cos\theta \sin^2\theta z) + 0$$

$$= \cos^4\theta z^2 + \cos\theta \sin^2\theta z + \cos^2\theta \sin^2\theta z^2 - \cos\theta \sin^2\theta z$$

$$= \cos^4\theta z^2 + \cos^2\theta \sin^2\theta z^2$$

$$= (\cos^4\theta + \cos^2\theta \sin^2\theta) z^2$$

$$= \cos^2\theta (\cos^2\theta + \sin^2\theta) z^2$$

$$= \cos^2\theta \cdot 1 \cdot z^2 = \cos^2\theta z^2$$

Setting Up the Integral

The flux integral becomes:

$$\Phi = \int_0^{\frac{\pi}{2}} \int_0^1 \cos^2\theta z^2 \, dz \, d\theta$$

Evaluating the Integral

1. **Integrate with respect to z:**

$$\int_0^1 z^2 \, dz = \left[\frac{z^3}{3} \right]_0^1 = \frac{1}{3}$$

Thus,

$$\Phi = \int_0^{\frac{\pi}{2}} \cos^2 \theta \cdot \frac{1}{3} \, d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

2. **Integrate with respect to θ:**
Recall the identity:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Thus,

$$\int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(2\theta) \, d\theta$$

$$= \frac{1}{2} [\theta]_0^{\frac{\pi}{2}} + \frac{1}{2} \left[\frac{\sin(2\theta)}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) + \frac{1}{4} (\sin \pi - \sin 0) = \frac{\pi}{4} + 0 = \frac{\pi}{4}$$

Thus,

$$\Phi = \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12}$$

4.5 Final Answer

The outward flux of the vector field $\mathbf{V} = \langle x^3z^2 + y^2z,\ x^2yz^2 - xyz,\ xz^4 - y^5 \rangle$ through the surface S is:

$$\boxed{\Phi = \frac{\pi}{12}}$$