Notes for 18.02 Recitation 21

18.02 Recitation MW9

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Mary: Wait a second. You can have anything you want, and you're asking for my phone number?
Carl: Yes.
Mary: 273-9164. Area code 415.

— *Sneakers (1992)*

This handout (and any other DLC's I write) are posted at `https://web.evanchen.cc/1802.html`.

§1 Definition of flux and accompanying shorthand

Definition of flux

Let $\mathbf{r}(u, v) : \mathcal{R} \to \mathbb{R}^3$ parametrize a surface $\mathcal{S}$ in $\mathbb{R}^3$. The flux of a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ through $\mathcal{S}$ is defined by

$$\iint_{\mathcal{R}} \mathbf{F}(\mathbf{r}(u, v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v.$$

We abbreviate $\mathbf{n} \, \mathrm{d}S := \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v$, so the flux integral becomes just $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$. In fact, people often split the shorthand $\mathbf{n} \, \mathrm{d}S$ into two parts:

$$\mathbf{n} := \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|} \quad \text{and} \quad \mathrm{d}S := \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u \, \mathrm{d}v.$$

(So that $\mathbf{n} \, \mathrm{d}S$ is indeed the full cross product, as the two absolute value things cancel.) That is, $\mathbf{n}$ is the unit vector in the *direction* of the cross product, while $\mathrm{d}S$ represents the *absolute value* of the cross product with $\mathrm{d}u \, \mathrm{d}v$ tacked on.

§2 Cross products are annoying, so we pre-compute them all

The cross product is so annoying to compute that in practice it's actually easier to just remember what it works out to in all the "common" 18.02 cases. That data is summarized below in the following table.

Derivations are written in Section 28.4 of my notes. Notice that you *only need the fifth column* for calculation. (And the third and fourth column can be deduced from the fifth one easily.) I think the only purpose of the third and fourth column is that for cylinder and sphere, it might be easier to remember both the third and fourth column and multiply them together.

Surface	Param's	$\mathbf{n}$ (unit vec)	$\mathrm{d}S$	$\mathbf{n}\,\mathrm{d}S = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v$
$z = f(x,y)$	(x,y)	$\frac{\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\rangle}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}$	$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$	$\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle \mathrm{d}x\,\mathrm{d}y$
$g(x,y,z) = c$	(x,y)	$\frac{\nabla g}{\lvert\nabla g\rvert}$	$\frac{\lvert\nabla g\rvert}{\lvert\partial g/\partial z\rvert}\,\mathrm{d}x\,\mathrm{d}y$	$\frac{\nabla g}{\partial g/\partial z}\,\mathrm{d}x\,\mathrm{d}y$
Cylindrical coords with fixed R	(θ, z)	$\langle \cos\theta, \sin\theta, 0\rangle$	$R\,\mathrm{d}\theta\,\mathrm{d}z$	$\langle R\cos\theta, R\sin\theta, 0\rangle\,\mathrm{d}\theta\,\mathrm{d}z$
Spherical coords with fixed R	(φ, θ)	$\frac{1}{R}\cdot\mathbf{r}(\varphi,\theta)$ (if $0 \le \varphi \le \pi$)	$R^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta$ (if $0 \le \varphi \le \pi$)	$R\sin\varphi\cdot\mathbf{r}(\varphi,\theta)\,\mathrm{d}\varphi\,\mathrm{d}\theta$

§3 Recipe (see LAMV 33.5 for slightly longer version)

Recipe for computing flux integrals with bare-hands parametrization (abridged)

To compute the flux of $\mathbf{F}$ over a surface $\mathcal{S}$:

1. Get the cross product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$.
   - If you are using (x,y)-coordinates to parametrize, use rows 1 or 2 of the table.
   - If $\mathcal{S}$ is specifically a cylinder or sphere, use rows 3 or 4 of the table.
   - Otherwise, parametrize $\mathbf{r}(u,v) : \mathcal{R} \to \mathbb{R}^3$ of the surface $\mathcal{S}$ and manually evaluate.
2. Look at which way the cross product points (via right-hand rule). Does it point "outward"? If not, negate the cross product (equivalently, swap the order of u and v) before going on.
3. Compute the dot product
$$\mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right).$$
   This gives you a number at every point on the parametrizing region $\mathcal{R}$.
4. Integrate the entire thing over $\mathcal{R}$ using any of the methods for double integrals (such as horizontal/vertical slicing, polar coordinates, change of variables, etc.).

§4 Recitation questions from official course

In this recitation, we do some examples of flux calculations for graphs, cylinders, and spheres. In each case, review the formula (and meaning) of $\mathbf{n}\,\mathrm{d}S$ before applying to the example at hand.

1. Let S be the portion of the cone $z = \sqrt{x^2+y^2}$ lying in the region $1 \le z \le 2$. Compute the upward flux of $\mathbf{V} = \langle x, y, 2z\rangle$ through S.

2. Find the flux of $\mathbf{V} = \langle z, x, y\rangle$ outward through the portion of the cylinder $x^2 + y^2 = 100$ in the first octant and below the plane $z = h$.

3. Find the outward flux of the vector field $\mathbf{V} = xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ through that part of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant $(x, y, z \ge 0)$.

4. Let S be the portion of the cylinder $x^2 + y^2 = 1$ in the octant $x, y, z \ge 0$ that lies below $z = 1$. Compute the outward flux of $\mathbf{V} = (x^3z^2 + y^2z)\mathbf{i} + (x^2yz^2 - xyz)\mathbf{j} + (xz^4 - y^5)\mathbf{k}$ through S.