Quick answer key to Recitation 20

ChatGPT 4o

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution

Consider the surface defined by $F(x, y, z) = 2$, where $F(x, y, z) = xyz$. We will address each part of the problem step by step.

1.1 Part (a): Finding a Parametrization

We aim to find a parametrization of the surface in the form:

$$\mathbf{r}(u, v) = (u, v, z(u, v))$$

such that $F(x, y, z) = 2$.

Given:

$$F(x, y, z) = xyz = 2$$

Substituting $\mathbf{r}(u, v)$ into the equation:

$$u \cdot v \cdot z(u, v) = 2 \implies z(u, v) = \frac{2}{uv}$$

Thus, the parametrization is:

$$\boxed{\mathbf{r}(u, v) = \left(u,\ v,\ \frac{2}{uv}\right)}$$

Domain Considerations: - $u \neq 0$ and $v \neq 0$ to avoid division by zero. - $uv > 0$ to ensure $z(u, v)$ is real and positive (since $F(x, y, z) = 2$ is positive).

1.2 Part (b): Computing the Tangent Vectors $\mathbf{r}_u$ and $\mathbf{r}_v$

Given the parametrization:

$$\mathbf{r}(u,v) = \left(u,\ v,\ \frac{2}{uv}\right)$$

we compute the partial derivatives with respect to u and v.

Tangent Vector $\mathbf{r}_u$:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial u}{\partial u},\ \frac{\partial v}{\partial u},\ \frac{\partial}{\partial u}\left(\frac{2}{uv}\right)\right) = \left(1,\ 0,\ -\frac{2}{u^2 v}\right)$$

Tangent Vector $\mathbf{r}_v$:

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial u}{\partial v},\ \frac{\partial v}{\partial v},\ \frac{\partial}{\partial v}\left(\frac{2}{uv}\right)\right) = \left(0,\ 1,\ -\frac{2}{uv^2}\right)$$

Thus, the tangent vectors are:

$$\boxed{\mathbf{r}_u = \left(1,\ 0,\ -\frac{2}{u^2 v}\right) \quad \text{and} \quad \mathbf{r}_v = \left(0,\ 1,\ -\frac{2}{uv^2}\right)}$$

1.3 Part (c): Equation of the Tangent Plane at $(2, 1, 1)$

To find the equation of the tangent plane at the point $(2, 1, 1)$, we can use the gradient of $F(x, y, z)$ since the surface is implicitly defined by $F(x, y, z) = 2$.

Gradient of $F(x, y, z)$:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \ \frac{\partial F}{\partial y}, \ \frac{\partial F}{\partial z} \right) = (yz, \ xz, \ xy)$$

At the point $(2, 1, 1)$:

$$\nabla F(2, 1, 1) = (1 \cdot 1, \ 2 \cdot 1, \ 2 \cdot 1) = (1, \ 2, \ 2)$$

Equation of the Tangent Plane: The equation of the tangent plane at (x_0, y_0, z_0) is given by:

$$\nabla F(x_0, y_0, z_0) \cdot ((x - x_0), \ (y - y_0), \ (z - z_0)) = 0$$

Substituting the known values:

$$(1, \ 2, \ 2) \cdot (x - 2, \ y - 1, \ z - 1) = 0 \implies 1(x - 2) + 2(y - 1) + 2(z - 1) = 0$$

Simplifying:

$$x - 2 + 2y - 2 + 2z - 2 = 0 \implies x + 2y + 2z - 6 = 0$$

Thus, the equation of the tangent plane is:

$$\boxed{x + 2y + 2z = 6}$$

1.4 Alternative Method for Part (c): Using Tangent Vectors

Alternatively, we can find the tangent plane using the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

Tangent Vectors at $(2, 1, 1)$: First, determine the corresponding parameters u and v for the point $(2, 1, 1)$:

$$x = u = 2, \quad y = v = 1, \quad z = \frac{2}{uv} = \frac{2}{2 \cdot 1} = 1$$

Thus, $u = 2$ and $v = 1$.

Substitute $u = 2$ and $v = 1$ into $\mathbf{r}_u$ and $\mathbf{r}_v$:

$$\mathbf{r}_u = \left(1,\ 0,\ -\frac{2}{(2)^2 \cdot 1}\right) = \left(1,\ 0,\ -\frac{1}{2}\right)$$

$$\mathbf{r}_v = \left(0,\ 1,\ -\frac{2}{2 \cdot (1)^2}\right) = (0,\ 1,\ -1)$$

Normal Vector to the Tangent Plane: The normal vector $\mathbf{n}$ is given by the cross product $\mathbf{r}_u \times \mathbf{r}_v$:

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i}\left(0 \cdot (-1) - (-\frac{1}{2}) \cdot 1\right) - \mathbf{j}\left(1 \cdot (-1) - (-\frac{1}{2}) \cdot 0\right) + \mathbf{k}\,(1 \cdot 1 - 0 \cdot 0)$$

$$= \mathbf{i}\left(0 + \frac{1}{2}\right) - \mathbf{j}\,(-1 - 0) + \mathbf{k}\,(1 - 0)$$

$$= \frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$$

To eliminate the fraction, multiply by 2:

$$\mathbf{n} = (1, 2, 2)$$

Equation of the Tangent Plane: Using the point-normal form of the plane equation:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{r}_0 = (2, 1, 1)$, we have:

$$1(x - 2) + 2(y - 1) + 2(z - 1) = 0$$

Simplifying:

$$x - 2 + 2y - 2 + 2z - 2 = 0 \implies x + 2y + 2z - 6 = 0$$

Thus, the equation of the tangent plane is:

$$\boxed{x + 2y + 2z = 6}$$

Both methods yield the same result, confirming the correctness of the tangent plane equation.

2 Problem: Surface of Revolution

Consider the surface S obtained by rotating the graph of $y = \sqrt{x}$, where $0 \leq x \leq 4$, around the x-axis.

(a) Parametrize this surface using x and θ, by writing $\mathbf{r}(x, \theta) = (x, y(x, \theta), z(x, \theta))$. Here, θ will be the angle of rotation.

(b) Find the cross product $\mathbf{r}_x \times \mathbf{r}_\theta$.

(c) Find the surface area of S.

2.1 Part (a): Parametrization of the Surface

To parametrize the surface obtained by rotating $y = \sqrt{x}$ around the x-axis, we introduce an angle θ representing the rotation around the x-axis.

For a fixed x in the interval $[0, 4]$, the corresponding y-coordinate on the curve is $y = \sqrt{x}$. When rotated by an angle θ, the point $(\sqrt{x}, 0)$ in the yz-plane traces a circle of radius $\sqrt{x}$.

Thus, the parametrization in terms of x and θ is:

$$\mathbf{r}(x, \theta) = \big(x,\ \sqrt{x}\cos\theta,\ \sqrt{x}\sin\theta\big)$$

where:

$$0 \le x \le 4, \quad 0 \le \theta < 2\pi$$

2.2 Part (b): Computing the Cross Product $\mathbf{r}_x \times \mathbf{r}_\theta$

First, we compute the partial derivatives of $\mathbf{r}(x, \theta)$ with respect to x and θ:

Partial Derivative with respect to x:

$$\mathbf{r}_x = \frac{\partial \mathbf{r}}{\partial x} = \left(\frac{\partial x}{\partial x}, \frac{\partial}{\partial x} \left(\sqrt{x} \cos \theta \right), \frac{\partial}{\partial x} \left(\sqrt{x} \sin \theta \right) \right) = \left(1, \frac{1}{2\sqrt{x}} \cos \theta, \frac{1}{2\sqrt{x}} \sin \theta \right)$$

Partial Derivative with respect to θ:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \left(\frac{\partial x}{\partial \theta}, \frac{\partial}{\partial \theta} \left(\sqrt{x} \cos \theta \right), \frac{\partial}{\partial \theta} \left(\sqrt{x} \sin \theta \right) \right) = \left(0, -\sqrt{x} \sin \theta, \sqrt{x} \cos \theta \right)$$

Cross Product $\mathbf{r}_x \times \mathbf{r}_\theta$: Using the determinant formula for the cross product:

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{1}{2\sqrt{x}} \cos \theta & \frac{1}{2\sqrt{x}} \sin \theta \\ 0 & -\sqrt{x} \sin \theta & \sqrt{x} \cos \theta \end{vmatrix}$$

Calculating the determinant:

$$\begin{aligned}
\mathbf{r}_x \times \mathbf{r}_\theta &= \mathbf{i} \left(\frac{1}{2\sqrt{x}} \cos \theta \cdot \sqrt{x} \cos \theta - \frac{1}{2\sqrt{x}} \sin \theta \cdot (-\sqrt{x} \sin \theta) \right) - \mathbf{j} \left(1 \cdot \sqrt{x} \cos \theta - \frac{1}{2\sqrt{x}} \sin \theta \cdot 0 \right) + \mathbf{k} \left(1 \cdot (- \right. \\
&= \mathbf{i} \left(\frac{\cos^2 \theta}{2} + \frac{\sin^2 \theta}{2} \right) - \mathbf{j} \left(\sqrt{x} \cos \theta \right) + \mathbf{k} \left(-\sqrt{x} \sin \theta \right) \\
&= \mathbf{i} \left(\frac{\cos^2 \theta + \sin^2 \theta}{2} \right) - \mathbf{j} \left(\sqrt{x} \cos \theta \right) - \mathbf{k} \left(\sqrt{x} \sin \theta \right) \\
&= \frac{1}{2} \mathbf{i} - \sqrt{x} \cos \theta \, \mathbf{j} - \sqrt{x} \sin \theta \, \mathbf{k}
\end{aligned}$$

Thus, the cross product is:

$$\boxed{\mathbf{r}_x \times \mathbf{r}_\theta = \frac{1}{2} \mathbf{i} - \sqrt{x} \cos \theta \, \mathbf{j} - \sqrt{x} \sin \theta \, \mathbf{k}}$$

2.3 Part (c): Finding the Surface Area of S

The surface area A of S can be computed using the surface integral:

$$A = \iint_S dS = \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta| \, dx \, d\theta$$

where D is the domain of the parameters x and θ.

Magnitude of the Cross Product:

$$|\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\sqrt{x}\cos\theta\right)^2 + \left(\sqrt{x}\sin\theta\right)^2} = \sqrt{\frac{1}{4} + x\cos^2\theta + x\sin^2\theta}$$

$$= \sqrt{\frac{1}{4} + x(\cos^2\theta + \sin^2\theta)} = \sqrt{\frac{1}{4} + x} = \sqrt{x + \frac{1}{4}}$$

Setting Up the Integral:

$$A = \int_{\theta=0}^{2\pi} \int_{x=0}^{4} \sqrt{x + \frac{1}{4}} \, dx \, d\theta$$

Evaluating the Integral: First, integrate with respect to x:

$$\int_0^4 \sqrt{x + \frac{1}{4}} \, dx$$

Let $u = x + \frac{1}{4}$, then $du = dx$ and the limits become $u = \frac{1}{4}$ to $u = 4 + \frac{1}{4} = \frac{17}{4}$.

Thus,

$$\int_0^4 \sqrt{x + \frac{1}{4}} \, dx = \int_{\frac{1}{4}}^{\frac{17}{4}} u^{1/2} \, du = \left[\frac{2}{3}u^{3/2}\right]_{\frac{1}{4}}^{\frac{17}{4}} = \frac{2}{3}\left(\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right)$$

$$= \frac{2}{3}\left(\frac{17^{3/2}}{8} - \frac{1^{3/2}}{8}\right) = \frac{2}{3} \cdot \frac{17^{3/2} - 1}{8} = \frac{17^{3/2} - 1}{12}$$

Next, integrate with respect to θ:

$$A = \int_0^{2\pi} \frac{17^{3/2} - 1}{12} \, d\theta = \frac{17^{3/2} - 1}{12} \cdot 2\pi = \frac{(17^{3/2} - 1)\pi}{6}$$

Simplify $17^{3/2}$:

$$17^{3/2} = \sqrt{17}^3 = 17\sqrt{17}$$

Thus,

$$A = \frac{(17\sqrt{17} - 1)\pi}{6}$$

Final Answer: The surface area of S is:

$$\boxed{A = \frac{(17\sqrt{17} - 1)\pi}{6}}$$

3 Flux of the Vector Field over the Unit Disk

We are tasked with calculating the flux of the vector field $\mathbf{V}(x, y, z) = \langle 1+z,\ 1,\ z^2 \rangle$ over the surface S, which is the unit disk on the xy-plane. The disk is oriented upward with the unit normal vector $\mathbf{n} = \mathbf{k} = \langle 0,\ 0,\ 1 \rangle$.

3.1 Understanding the Surface and the Vector Field

- **Surface S:** - Defined by $z = 0$ (since it lies on the xy-plane). - Bounded by $x^2 + y^2 \leq 1$ (unit disk). - Oriented upward with normal vector $\mathbf{n} = \mathbf{k}$.

- **Vector Field $\mathbf{V}$:** - Given by $\mathbf{V}(x, y, z) = \langle 1 + z,\ 1,\ z^2 \rangle$. - Components: - $V_x = 1 + z$ - $V_y = 1$ - $V_z = z^2$

3.2 Calculating the Flux

The flux Φ of the vector field $\mathbf{V}$ across the surface S is given by the surface integral:

$$\Phi = \iint_S \mathbf{V} \cdot \mathbf{n} \, dS$$

where: - $\mathbf{n}$ is the unit normal vector to the surface S. - dS is the differential element of the surface area.

Evaluating $\mathbf{V} \cdot \mathbf{n}$

Given $\mathbf{n} = \mathbf{k} = \langle 0, 0, 1 \rangle$, the dot product $\mathbf{V} \cdot \mathbf{n}$ simplifies to:

$$\mathbf{V} \cdot \mathbf{n} = V_x \cdot 0 + V_y \cdot 0 + V_z \cdot 1 = V_z$$

Substituting $V_z = z^2$:

$$\mathbf{V} \cdot \mathbf{n} = z^2$$

Simplifying on Surface S

On the surface S (the unit disk on the xy-plane), the z-coordinate is zero:

$$z = 0$$

Therefore:

$$\mathbf{V} \cdot \mathbf{n} = z^2 = 0^2 = 0$$

3.3 Conclusion

Since $\mathbf{V} \cdot \mathbf{n} = 0$ everywhere on the surface S, the flux Φ is zero:

$$\Phi = \iint_S \mathbf{V} \cdot \mathbf{n} \, dS = \iint_S 0 \, dS = 0$$

3.4 Final Answer

The flux of $\mathbf{V}$ across the unit disk S is:

$$\boxed{0}$$

4 Center of Mass and Flux Calculations

4.1 Problem Statement

Let S be the sphere of radius 2 centered at $(0, 0, 0)$.

(a) Show that the outward unit normal vector $\mathbf{n}$ at the point $(x, y, z) \in S$ is given by $\frac{1}{2}\langle x, y, z\rangle$ using the description of S as a level surface.

(b) Calculate (without doing any integration) the outward flux $\iint_S \mathbf{V} \cdot \mathbf{n}\, dS$ of the vector field $\mathbf{V}(x, y, z) = \langle x, y, z\rangle$.

4.2 Part (a): Determining the Outward Unit Normal Vector

To find the outward unit normal vector $\mathbf{n}$ at a point (x, y, z) on the sphere S, we can utilize the concept of level surfaces and gradients.

Step 1: Expressing the Sphere as a Level Surface

The sphere S of radius 2 centered at the origin can be described by the equation:

$$F(x, y, z) = x^2 + y^2 + z^2 = 4$$

This represents a level surface where $F(x, y, z) = 4$.

Step 2: Calculating the Gradient of F

The gradient of F, denoted ∇F, points in the direction of the greatest rate of increase of F and is normal to the surface S:

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \langle 2x, 2y, 2z \rangle$$

Step 3: Determining the Unit Normal Vector

The unit normal vector $\mathbf{n}$ is obtained by normalizing the gradient:

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{\langle 2x, 2y, 2z \rangle}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = \frac{\langle 2x, 2y, 2z \rangle}{2\sqrt{x^2 + y^2 + z^2}}$$

Since (x, y, z) lies on the sphere S, $x^2 + y^2 + z^2 = 4$, hence:

$$\mathbf{n} = \frac{\langle 2x, 2y, 2z \rangle}{2 \cdot 2} = \frac{1}{2}\langle x, y, z \rangle$$

Thus, the outward unit normal vector at any point (x, y, z) on S is:

$$\boxed{\mathbf{n} = \frac{1}{2}\langle x, y, z \rangle}$$

4.3 Part (b): Calculating the Outward Flux

We are to compute the outward flux of the vector field $\mathbf{V}(x, y, z) = \langle x, y, z\rangle$ across the sphere S. The flux is given by:

$$\Phi = \iint_S \mathbf{V} \cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is the outward unit normal vector obtained in Part (a).

Step 1: Computing $\mathbf{V} \cdot \mathbf{n}$

Substitute $\mathbf{n} = \frac{1}{2}\langle x, y, z\rangle$ into the dot product:

$$\mathbf{V} \cdot \mathbf{n} = \langle x, y, z\rangle \cdot \frac{1}{2}\langle x, y, z\rangle = \frac{1}{2}(x^2 + y^2 + z^2)$$

On the sphere S, $x^2 + y^2 + z^2 = 4$, so:

$$\mathbf{V} \cdot \mathbf{n} = \frac{1}{2} \times 4 = 2$$

Thus, the integrand is a constant:

$$\mathbf{V} \cdot \mathbf{n} = 2$$

Step 2: Evaluating the Flux Integral

Since the integrand is constant, the flux simplifies to:

$$\Phi = \iint_S 2\, dS = 2 \iint_S dS = 2 \cdot \text{Surface Area of } S$$

The surface area A of a sphere with radius 2 is:

$$A = 4\pi(2)^2 = 16\pi$$

Therefore, the flux is:

$$\Phi = 2 \times 16\pi = 32\pi$$

Thus, the outward flux of $\mathbf{V}$ across S is:

$$\boxed{\Phi = 32\pi}$$

Alternative Approach: Using the Divergence Theorem

For verification, we can also apply the Divergence Theorem, which states:

$$\iint_S \mathbf{V} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{V} \, dV$$

where V is the volume enclosed by S.

Calculating the Divergence of V:

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 1 + 1 + 1 = 3$$

Calculating the Volume of V: The volume V of the sphere with radius 2 is:

$$V = \frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi$$

Applying the Divergence Theorem:

$$\Phi = \iiint_V 3 \, dV = 3 \times \frac{32}{3}\pi = 32\pi$$

This confirms our previous result:

$$\boxed{\Phi = 32\pi}$$