Quick answer key to Recitation 19

ChatGPT 4o

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1 Solution

We are tasked with expressing the triple integral

$$\iiint_R f \, dV$$

as iterated integrals in both spherical and cylindrical coordinates, where R is the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the upper hemisphere of radius 2, defined by $x^2 + y^2 + z^2 \leq 4$ with $z \geq 0$.

1.1 Understanding the Region R

1.1.1 Description of R

- **Lower Bound:** The cone $z = \sqrt{x^2 + y^2}$ can be rewritten as $z^2 = x^2+y^2$, representing a double-napped cone. Since $z \geq 0$, we consider only the upper nappe of the cone. - **Upper Bound:** The upper hemisphere $x^2+y^2+z^2 = 4$ with $z \geq 0$. - **Intersection:** To find the limits of integration, determine where the cone and the hemisphere intersect.

1.1.2 Finding the Intersection of the Cone and Hemisphere

Set the equations equal to find the boundary of R:

$$z^2 = x^2 + y^2 \quad \text{and} \quad x^2 + y^2 + z^2 = 4$$

Substitute $z^2 = x^2 + y^2$ into the hemisphere equation:

$$x^2 + y^2 + (x^2 + y^2) = 42(x^2 + y^2) = 4x^2 + y^2 = 2$$

Thus, the cone and the hemisphere intersect along the circle $x^2 + y^2 = 2$ and $z = \sqrt{2}$.

1.2 Part (a): Expressing the Integral in Spherical Coordinates

1.2.1 Spherical Coordinates Overview

Spherical coordinates (ρ, θ, ϕ) are related to Cartesian coordinates by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

where:

- $\rho \geq 0$ is the radial distance,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- $0 \leq \phi \leq \pi$ is the polar angle.

The volume element in spherical coordinates is:

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

1.2.2 Determining the Limits of Integration

- **Radial Distance (ρ):** The hemisphere has a radius of 2, so $0 \leq \rho \leq 2$.
- **Azimuthal Angle (θ):** Full rotation around the z-axis, so $0 \leq \theta < 2\pi$.
- **Polar Angle (ϕ):** Determined by the cone $z = \sqrt{x^2 + y^2}$.
  The cone equation in spherical coordinates:

$$\rho \cos\phi = \rho \sin\phi \cos\phi = \sin\phi \tan\phi = 1\phi = \frac{\pi}{4}$$

Therefore, ϕ ranges from 0 to $\frac{\pi}{4}$.

1.2.3 Setting Up the Integral

The triple integral in spherical coordinates becomes:

$$\iiint_R f \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 f(\rho, \theta, \phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

where $f(\rho, \theta, \phi)$ is the function expressed in spherical coordinates.

1.3 Part (b): Expressing the Integral in Cylindrical Coordinates

1.3.1 Cylindrical Coordinates Overview

Cylindrical coordinates (r, θ, z) are related to Cartesian coordinates by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where:

- $r \geq 0$ is the radial distance,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- z is the height.

The volume element in cylindrical coordinates is:

$$dV = r\,dz\,dr\,d\theta$$

1.3.2 Determining the Limits of Integration

- **Radial Distance (r):** The region is bounded by the cone and the hemisphere. At any height z, the radial distance r satisfies:

From the cone $z = r$, and from the hemisphere $r^2 + z^2 = 4$, so $r = \sqrt{4 - z^2}$.

- **Height (z):** The cone starts at $z = 0$ and goes up to the intersection point $z = \sqrt{2}$.

- **Azimuthal Angle (θ):** Full rotation around the z-axis, so $0 \leq \theta < 2\pi$.

Thus, the limits are:

$$0 \leq \theta < 2\pi 0 \leq z \leq \sqrt{2} z \leq r \leq \sqrt{4 - z^2}$$

1.3.3 Setting Up the Integral

The triple integral in cylindrical coordinates becomes:

$$\iiint_R f\,dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_z^{\sqrt{4-z^2}} f(r, \theta, z)\, r\,dr\,dz\,d\theta$$

where $f(r, \theta, z)$ is the function expressed in cylindrical coordinates.

1.4 Conclusion

The triple integral $\iiint_R f\,dV$ over the region R can be expressed as iterated integrals in both spherical and cylindrical coordinates as follows:

(a) **Spherical Coordinates**:

$$\iiint_R f\,dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) **Cylindrical Coordinates**:

$$\iiint_R f\,dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_z^{\sqrt{4-z^2}} f(r, \theta, z)\, r \, dr \, dz \, d\theta$$

These iterated integrals allow for the evaluation of the triple integral over the specified region R using the most convenient coordinate system based on the symmetry of the region and the function f.

2 Center of Mass of a Hemisphere

We are tasked with finding the center of mass of a solid hemisphere of radius a using spherical coordinates. The hemisphere is assumed to have a uniform density $\delta = 1$.

2.1 Understanding the Problem

Consider the upper hemisphere defined by:

$$z \geq 0 \quad \text{and} \quad x^2 + y^2 + z^2 \leq a^2.$$

Due to the symmetry of the hemisphere about the z-axis, the coordinates of the center of mass will satisfy:

$$\overline{x} = 0, \quad \overline{y} = 0.$$

Thus, we only need to determine the z-coordinate of the center of mass, $\overline{z}$.

2.2 Formula for Center of Mass

The z-coordinate of the center of mass for a solid region R with density $\delta = 1$ is given by:

$$\overline{z} = \frac{1}{M} \iiint_R z \, dV,$$

where M is the mass of the hemisphere:

$$M = \iiint_R dV.$$

2.3 Setting Up the Integral in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are related to Cartesian coordinates by:

$$x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r \cos\phi,$$

where:

- $r \geq 0$ is the radial distance,
- $0 \leq \theta < 2\pi$ is the azimuthal angle,
- $0 \leq \phi \leq \pi$ is the polar angle.

The volume element in spherical coordinates is:

$$dV = r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

2.3.1 Limits of Integration

For the upper hemisphere:

$$0 \leq r \leq a, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

2.4 Calculating the Mass M

First, compute the mass M:

$$M = \iiint_R dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

2.4.1 Evaluating the Integral

1. **Integrate with respect to r:**

$$\int_0^a r^2 \, dr = \left[\frac{r^3}{3}\right]_0^a = \frac{a^3}{3}.$$

2. **Integrate with respect to ϕ:**

$$\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi = [-\cos\phi]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = 0 + 1 = 1.$$

3. **Integrate with respect to θ:**

$$\int_0^{2\pi} d\theta = 2\pi.$$

4. **Combine the results:**

$$M = \frac{a^3}{3} \times 1 \times 2\pi = \frac{2\pi a^3}{3}.$$

2.5 Calculating triple integral

Next, compute the integral $\iiint_R z\, dV$:

$$\iiint_R z\, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a (r\cos\phi) r^2 \sin\phi\, dr\, d\phi\, d\theta.$$

2.5.1 Simplifying the Integrand

$$z\, dV = (r\cos\phi) \cdot r^2 \sin\phi\, dr\, d\phi\, d\theta = r^3 \cos\phi \sin\phi\, dr\, d\phi\, d\theta.$$

2.5.2 Evaluating the Integral

1. **Integrate with respect to r:**

$$\int_0^a r^3\, dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}.$$

2. **Integrate with respect to ϕ:**

$$\int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\, d\phi.$$

Use the substitution $u = \sin\phi$, $du = \cos\phi\, d\phi$:

$$\int_0^{\frac{\pi}{2}} u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

3. **Integrate with respect to θ:**

$$\int_0^{2\pi} d\theta = 2\pi.$$

4. **Combine the results:**

$$\iiint_R z\, dV = \frac{a^4}{4} \times \frac{1}{2} \times 2\pi = \frac{a^4\pi}{4}.$$

2.6 Calculating the Center of Mass $\overline{z}$

Using the formula:

$$\overline{z} = \frac{1}{M}\iiint_R z\,dV = \frac{\frac{a^4\pi}{4}}{\frac{2\pi a^3}{3}} = \frac{a^4\pi}{4} \times \frac{3}{2\pi a^3} = \frac{3a}{8}.$$

2.7 Conclusion

The center of mass of the hemisphere of radius a with uniform density is located at:

$$(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{3a}{8}\right).$$

3 Gravitational Attraction of a Region R on a Unit Mass at the Origin

We are tasked with finding the gravitational attraction of the region R bounded above by the plane $z = 2$ and below by the cone $z^2 = 4(x^2 + y^2)$, on a unit mass located at the origin. The region R has a constant density $\delta = 1$.

3.1 Understanding the Region R

Description of R

- **Lower Bound:** The cone $z^2 = 4(x^2 + y^2)$ can be rewritten as $z = 2\sqrt{x^2 + y^2}$ (considering $z \geq 0$). - **Upper Bound:** The plane $z = 2$. - **Intersection:** To find the boundary of R, set $z = 2\sqrt{x^2 + y^2}$ equal to $z = 2$:

$$2\sqrt{x^2 + y^2} = 2 \implies \sqrt{x^2 + y^2} = 1 \implies x^2 + y^2 = 1$$

Thus, the cone and the plane intersect along the circle $x^2 + y^2 = 1$ at $z = 2$.

3.2 Setting Up the Gravitational Attraction

The gravitational attraction $\mathbf{F}$ at the origin due to the mass distribution in R is given by:

$$\mathbf{F} = -G \iiint_R \frac{\mathbf{r}}{|\mathbf{r}|^3} \delta \, dV$$

where: - G is the gravitational constant (assuming $G = 1$ for simplicity), - $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of a point in R, - $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Due to the symmetry of the region R about the z-axis, the x and y-components of $\mathbf{F}$ will cancel out, leaving only the z-component. Therefore, we focus on calculating the z-component of $\mathbf{F}$, denoted as F_z:

$$F_z = -G \iiint_R \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \, dV$$

Assuming $G = 1$, we have:

$$F_z = - \iiint_R \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \, dV$$

3.3 Converting to Cylindrical Coordinates

To evaluate the integral, we convert to cylindrical coordinates (r, θ, z), where:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

The volume element in cylindrical coordinates is:

$$dV = r\,dz\,dr\,d\theta$$

The integrand becomes:

$$\frac{z}{(r^2 + z^2)^{3/2}}$$

Determining the Limits of Integration

- **Radial Distance r:** For a fixed z, r ranges from 0 up to where the cone and plane intersect:

$$z = 2r \implies r = \frac{z}{2}$$

However, since the plane $z = 2$ bounds z, r ranges from 0 to $\frac{z}{2}$.

- **Height z:** Ranges from the base of the cone $z = 0$ up to the plane $z = 2$.

- **Azimuthal Angle θ:** Full rotation around the z-axis, $0 \leq \theta < 2\pi$.

Thus, the limits are:

$$0 \leq \theta < 2\pi, \quad 0 \leq z \leq 2, \quad 0 \leq r \leq \frac{z}{2}$$

3.4 Setting Up the Integral

The z-component of the gravitational attraction is:

$$F_z = -\int_0^{2\pi} \int_0^{2} \int_0^{\frac{z}{2}} \frac{z}{(r^2 + z^2)^{3/2}} \cdot r \, dr \, dz \, d\theta$$

3.5 Evaluating the Integral

Step 1: Integrate with Respect to r

Consider the inner integral:

$$I_r = \int_0^{\frac{z}{2}} \frac{z \cdot r}{(r^2 + z^2)^{3/2}} \, dr$$

Let $u = r^2 + z^2$, then $du = 2r\,dr$, so $r\,dr = \frac{du}{2}$.

Substituting:

$$I_r = z \cdot \frac{1}{2} \int_{u=z^2}^{u=z^2+\left(\frac{z}{2}\right)^2} u^{-3/2} \, du = \frac{z}{2} \left[-2u^{-1/2} \right]_{z^2}^{\frac{5z^2}{4}} = \frac{z}{2} \left(-2 \cdot \frac{1}{\sqrt{\frac{5z^2}{4}}} + 2 \cdot \frac{1}{\sqrt{z^2}} \right)$$

Simplify:

$$I_r = \frac{z}{2} \left(-2 \cdot \frac{2}{z\sqrt{5}} + 2 \cdot \frac{1}{z} \right) = \frac{z}{2} \left(-\frac{4}{z\sqrt{5}} + \frac{2}{z} \right) = \frac{z}{2} \cdot \frac{-4 + 2\sqrt{5}}{z\sqrt{5}} = \frac{-4 + 2\sqrt{5}}{2\sqrt{5}} = \frac{-2 + \sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5} - 2}{\sqrt{5}}$$

Step 2: Integrate with Respect to z

Now, the integral becomes:

$$F_z = -\int_0^{2\pi} \int_0^2 \frac{\sqrt{5} - 2}{\sqrt{5}} \, dz \, d\theta = -\frac{\sqrt{5} - 2}{\sqrt{5}} \int_0^{2\pi} \int_0^2 dz \, d\theta$$

Evaluate the integrals:

$$\int_0^2 dz = 2, \quad \int_0^{2\pi} d\theta = 2\pi$$

Thus:

$$F_z = -\frac{\sqrt{5} - 2}{\sqrt{5}} \cdot 2 \cdot 2\pi = -4\pi \cdot \frac{\sqrt{5} - 2}{\sqrt{5}} = -4\pi \left(\frac{\sqrt{5} - 2}{\sqrt{5}} \right)$$

3.6 Simplifying the Expression

Rationalize the denominator:

$$F_z = -4\pi\left(\frac{\sqrt{5}-2}{\sqrt{5}}\right) = -4\pi\left(\frac{(\sqrt{5}-2)}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}\right) = -4\pi\left(\frac{5-2\sqrt{5}}{5}\right) = -4\pi\left(1-\frac{2\sqrt{5}}{5}\right) = -4\pi+\frac{8\pi}{\sqrt{5}}$$

Alternatively, leaving it in the original form is also acceptable:

$$F_z = -4\pi\left(1-\frac{2}{\sqrt{5}}\right)$$

3.7 Final Answer

The gravitational attraction of the region R on a unit mass at the origin is directed along the negative z-axis and has a magnitude of:

$$\mathbf{F} = \left(0, 0, -4\pi \left(1 - \frac{2}{\sqrt{5}}\right)\right)$$

Alternatively, rationalized:

$$\mathbf{F} = \left(0, 0, -4\pi \left(\frac{\sqrt{5} - 2}{\sqrt{5}}\right)\right)$$

This vector represents the gravitational force exerted by the region R on the unit mass at the origin.