Notes for 18.02 Recitation 19

18.02 Recitation MW9

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Alice: I'm going to start moving towards the door to start people moving by gravity.
Bob: Must be nice to be attractive.
Alice: I think you just called me massive.
Bob: Some people can't take a compliment.

This handout (and any other DLC's I write) are posted at `https://web.evanchen.cc/1802.html`.

§1 Spherical coordinates

In the mathematical notation, the spherical coordinate transition map $(\rho, \varphi, \theta) \mapsto (x, y, z)$ is defined by

$$\begin{aligned} x &= \rho \sin \varphi \cos \theta \\ y &= \rho \sin \varphi \sin \theta \\ z &= \rho \cos \varphi. \end{aligned}$$

(Again, different books have different conventions.) See LAMV §26.4 for a full derivation of this and figure and the corresponding Jacobian:

$$\mathrm{d}V := \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2 \sin \varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

§2 Volume

Integrate $1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

§3 Center of mass

Note we use δ instead of ρ to avoid conflicting with spherical coordinates.

Same deal: for a region $\mathcal{R}$ the center of mass is the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\begin{aligned} \bar{x} &:= \frac{1}{\operatorname{mass}(\mathcal{R})} \iiint_{\mathcal{R}} x\delta(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ \bar{y} &:= \frac{1}{\operatorname{mass}(\mathcal{R})} \iiint_{\mathcal{R}} y\delta(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ \bar{z} &:= \frac{1}{\operatorname{mass}(\mathcal{R})} \iiint_{\mathcal{R}} z\delta(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ \operatorname{mass}(\mathcal{R}) &:= \iiint_{\mathcal{R}} \delta(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

Unsurprisingly, when $\delta = 1$ is constant unit density, then mass equals volume.

§4 Gravity

Suppose a mass of point m is located at the origin $O = (0,0,0)$, In general, given a mass m at a point O and a point of mass M at a point P, Newton's law says the gravitational force exerted by P on O is

$$\mathbf{F}_{\text{gravity}} = \frac{GmM}{|OP|^2} \cdot \underbrace{\frac{\overrightarrow{OP}}{|OP|}}_{\text{unit vector from } O \text{ to } P}$$

where $G \approx 6.67408 \cdot 10^{-11} \cdot \mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$ is the gravitational constant.

But in real life, we usually want our mass M to be take up a whole region $\mathcal{R}$, with some density δ. (Point masses don't occur in real life unless you count black holes.) So let's suppose we have a solid mass occupying region $\mathcal{R}$. In that case, each individual point $P = (x,y,z)$ in $\mathcal{R}$ can be thought of as contributing

$$\frac{Gm \cdot (\delta(x,y,z)\,\mathrm{d}V)}{x^2+y^2+z^2} \cdot \underbrace{\frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}}_{\text{unit vector from } O \text{ to } P}.$$

Integrating over all of $\mathcal{R}$ gives a gravitational vector $\mathbf{G} = \langle G_1, G_2, G_3 \rangle$ defined by

$$\begin{aligned} G_1 &:= Gm \iiint_{\mathcal{R}} \frac{x\delta(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ G_2 &:= Gm \iiint_{\mathcal{R}} \frac{y\delta(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ G_3 &:= Gm \iiint_{\mathcal{R}} \frac{z\delta(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

That is, $\mathbf{G} = G_1\mathbf{e}_1 + G_2\mathbf{e}_2 + G_3\mathbf{e}_3$. Because the $(x^2+y^2+z^2)^{\frac{3}{2}}$ is so awkward to work with, you will commonly switch to spherical coordinates so that

$$(x^2+y^2+z^2)^{\frac{3}{2}} = \rho^3.$$

For example, the integrand for G_3 would be

$$\begin{aligned} \frac{z\delta(x,y,z)}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \frac{(\rho\cos\varphi)\delta(x,y,z)}{\rho^3}(\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta) \\ &= \delta(x,y,z)\sin\varphi\cos\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta. \end{aligned}$$

(Note the total mass M of the region $\mathcal{R}$ does not appear in these formulas!)

§5 Recitation questions from the official course

1. Express the triple integral $\iiint_R f dV$ as iterated integrals for the region R below the upper hemisphere of radius 2: $z^2 \le 4 - x^2 - y^2$ and above the cone $z = \sqrt{x^2+y^2}$ in (a) spherical coordinates; (b) cylindrical coordinates

2. Find the center of mass of a hemisphere of radius a, using spherical coordinates. Assume the density $\delta = 1$.

3. Find the gravitational attraction of the region R bounded above by the plane $z = 2$ and below by the cone $z^2 = 4(x^2+y^2)$,on a unit mass at the origin. Assume R has constant density $\delta = 1$.