Quick answer key to Recitation 18

ChatGPT 4o

November 17, 2024

Use the table of contents below to skip to a specific part without seeing spoilers to the other parts.

I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution

We are tasked with finding the volume of the region T bounded by the surface $x = y^2$, the plane $z = 0$, and the plane $x + z = 1$. Additionally, we will sketch T and describe its projections onto the xy-plane, xz-plane, and yz-plane.

1.1 Step 1: Understanding the Region T

1.1.1 Description of T

- The surface $x = y^2$ is a parabolic cylinder that opens along the positive x-axis. - The plane $z = 0$ is the xy-plane. - The plane $x + z = 1$ can be rewritten as $z = 1 - x$, which slopes downward from $z = 1$ at $x = 0$ to $z = 0$ at $x = 1$.

The region T is thus the set of points (x, y, z) such that:

$$y^2 \leq x \leq 1 \quad \text{and} \quad 0 \leq z \leq 1 - x$$

1.2 Step 2: Describing the Projections of T

1.2.1 Projection onto the xy-Plane

The projection onto the xy-plane involves ignoring the z-coordinate. The bounds are:

$$y^2 \leq x \leq 1 \quad \text{for} \quad y \in [-1, 1]$$

This projection is the region between the parabola $x = y^2$ and the vertical line $x = 1$.

1.2.2 Projection onto the xz-Plane

The projection onto the xz-plane involves ignoring the y-coordinate. The bounds are:

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq z \leq 1 - x$$

This projection is a right triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, 1)$.

1.2.3 Projection onto the yz-Plane

The projection onto the yz-plane involves ignoring the x-coordinate. Since $x \geq y^2$ and $x \leq 1$, the bounds are:

$$y^2 \leq 1 \implies y \in [-1, 1] \quad \text{and} \quad 0 \leq z \leq 1$$

This projection is a rectangle with $y \in [-1, 1]$ and $z \in [0, 1]$.

1.3 Step 3: Calculating the Volume Using Vertical Slicing

We will use the projection onto the xy-plane to set up the double integral for the volume.

1.3.1 Setting Up the Integral

The volume V can be expressed as:

$$V = \iint_R (z_{\text{upper}} - z_{\text{lower}}) \, dA$$

where:

$$z_{\text{upper}} = 1 - x \quad \text{and} \quad z_{\text{lower}} = 0$$

Thus:

$$V = \iint_R (1 - x) \, dA$$

The region R in the xy-plane is bounded by $y^2 \leq x \leq 1$ and $y \in [-1, 1]$.

1.3.2 Expressing the Integral in Terms of y and x

The integral becomes:

$$V = \int_{y=-1}^{1} \int_{x=y^2}^{1} (1 - x)\, dx\, dy$$

1.3.3 Evaluating the Inner Integral

First, integrate with respect to x:

$$\int_{x=y^2}^{1} (1 - x)\, dx = \left[x - \frac{x^2}{2} \right]_{y^2}^{1} = \left(1 - \frac{1}{2} \right) - \left(y^2 - \frac{y^4}{2} \right) = \frac{1}{2} - y^2 + \frac{y^4}{2}$$

1.3.4 Evaluating the Outer Integral

Now, integrate with respect to y:

$$V = \int_{y=-1}^{1} \left(\frac{1}{2} - y^2 + \frac{y^4}{2} \right) dy$$

Since the integrand is an even function (symmetric about the y-axis), we can simplify the computation:

$$V = 2 \int_{y=0}^{1} \left(\frac{1}{2} - y^2 + \frac{y^4}{2} \right) dy$$

Compute the integral:

$$\begin{aligned}
V &= 2 \left[\frac{1}{2} y - \frac{y^3}{3} + \frac{y^5}{10} \right]_0^1 \\
&= 2 \left(\frac{1}{2}(1) - \frac{1}{3}(1) + \frac{1}{10}(1) - 0 \right) \\
&= 2 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) \\
&= 2 \left(\frac{15}{30} - \frac{10}{30} + \frac{3}{30} \right) \\
&= 2 \left(\frac{8}{30} \right) \\
&= \frac{16}{30} \\
&= \frac{8}{15}
\end{aligned}$$

1.4 Conclusion

The volume of the region T bounded by the surface $x = y^2$, the plane $z = 0$, and the plane $x + z = 1$ is:

$$V = \frac{8}{15}$$

2 Solution

We are tasked with evaluating the triple integral

$$\iiint_R (2 - 2z)\, dV$$

where R is the pyramid with a square base defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ in the xy-plane, and a vertex at $(0, 0, 1)$.

2.1 Step 1: Sketching the Region R

- **Base:** Square in the xy-plane with vertices at $(-1, -1, 0)$, $(-1, 1, 0)$, $(1, 1, 0)$, and $(1, -1, 0)$. - **Vertex:** Point at $(0, 0, 1)$. - **Faces:** Four triangular faces connecting each edge of the base to the vertex.

2.2 Step 2: Describing the Cross Sections for Fixed z

For a fixed value of z between 0 and 1, the cross section of the pyramid R at height z is a square that shrinks linearly as z increases from 0 to 1.

2.2.1 Cross Section at Height z

- **Side Length:** At height z, the side length of the square cross section is $2(1 - z)$. - **Limits for x and y:**

$$-(1 - z) \leq x \leq 1 - z \quad \text{and} \quad -(1 - z) \leq y \leq 1 - z$$

2.3 Step 3: Setting Up the Triple Integral

Using the description of the cross sections, we can express the volume integral in Cartesian coordinates by integrating with respect to x, then y, and finally z.

2.3.1 Limits of Integration

$$\begin{cases} 0 \leq z \leq 1 \\ -(1-z) \leq x \leq 1-z \\ -(1-z) \leq y \leq 1-z \end{cases}$$

2.3.2 Expressing the Integral

$$\iiint_R (2-2z)\, dV = \int_{z=0}^{1} \int_{x=-(1-z)}^{1-z} \int_{y=-(1-z)}^{1-z} (2-2z)\, dy\, dx\, dz$$

2.4 Step 4: Evaluating the Integral

2.4.1 Integrate with Respect to y

$$\int_{y=-(1-z)}^{1-z} (2-2z)\,dy = (2-2z)\,[y]_{-(1-z)}^{1-z} = (2-2z)\left((1-z)-(-(1-z))\right) = (2-2z)(2(1-z)) = 4(1-z)(1$$

2.4.2 Integrate with Respect to x

$$\int_{x=-(1-z)}^{1-z} 4(1-z)^2\,dx = 4(1-z)^2\,[x]_{-(1-z)}^{1-z} = 4(1-z)^2\left((1-z)-(-(1-z))\right) = 4(1-z)^2 \cdot 2(1-z) = 8(1-z$$

2.4.3 Integrate with Respect to z

$$\int_{z=0}^{1} 8(1-z)^3\,dz$$

Let $u = 1-z$, then $du = -dz$. Changing the limits:

$$z = 0 \Rightarrow u = 1z = 1 \Rightarrow u = 0$$

Thus,

$$\int_{u=1}^{0} 8u^3(-du) = \int_{u=0}^{1} 8u^3\,du = 8\left[\frac{u^4}{4}\right]_0^1 = 8\left(\frac{1}{4}-0\right) = 2$$

2.5 Conclusion

The value of the triple integral is:

$$\iiint_R (2-2z)\,dV = 2$$

3 Solution

We are tasked with evaluating the triple integral

$$\iiint_D y^2 \, dV,$$

where D is the region defined by:

$$x^2 + y^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad z^2 \leq 4x^2 + 4y^2.$$

3.1 Step 1: Understanding the Region D

3.1.1 Description of D

- **Base:** The inequality $x^2 + y^2 \leq 1$ represents a cylinder of radius 1 centered along the z-axis. - **Lower Bound:** $z \geq 0$ confines the region to above the xy-plane. - **Upper Bound:** $z^2 \leq 4x^2 + 4y^2$ can be rewritten as $z \leq 2\sqrt{x^2 + y^2}$, representing a double cone opening upwards and downwards. However, since $z \geq 0$, only the upper half-cone is relevant.

3.2 Step 2: Converting to Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are defined by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

where $r \geq 0$ and $\theta \in [0, 2\pi)$.

3.2.1 Expressing the Bounds in Cylindrical Coordinates

- **Cylinder $x^2 + y^2 \leq 1$:**

$$r^2 \leq 1 \implies r \leq 1.$$

- **Cone $z = 2\sqrt{x^2 + y^2}$:**

$$z = 2r.$$

- **Lower Bound $z = 0$:**

$$z \geq 0.$$

3.2.2 Region D in Cylindrical Coordinates

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 2r.$$

3.3 Step 3: Setting Up the Triple Integral

The triple integral in cylindrical coordinates is:

$$\iiint_D y^2\, dV = \int_0^{2\pi} \int_0^1 \int_0^{2r} (r\sin\theta)^2 \cdot r\, dz\, dr\, d\theta.$$

Here, $y = r\sin\theta$ and $dV = r\, dz\, dr\, d\theta$.

3.4 Step 4: Evaluating the Integral

3.4.1 Simplifying the Integrand

$$y^2 = (r\sin\theta)^2 = r^2\sin^2\theta.$$

Thus, the integrand becomes:

$$y^2 \cdot r = r^2\sin^2\theta \cdot r = r^3\sin^2\theta.$$

The integral simplifies to:

$$\iiint_D y^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^3\sin^2\theta\,dz\,dr\,d\theta.$$

3.4.2 Integrating with Respect to z

$$\int_0^{2r} r^3\sin^2\theta\,dz = r^3\sin^2\theta\cdot[z]_0^{2r} = r^3\sin^2\theta\cdot 2r = 2r^4\sin^2\theta.$$

The integral becomes:

$$\iiint_D y^2\,dV = 2\int_0^{2\pi}\int_0^1 r^4\sin^2\theta\,dr\,d\theta.$$

3.4.3 Integrating with Respect to r

$$\int_0^1 r^4\,dr = \left[\frac{r^5}{5}\right]_0^1 = \frac{1}{5}.$$

Thus, the integral simplifies to:

$$\iiint_D y^2\,dV = 2\cdot\frac{1}{5}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{2}{5}\int_0^{2\pi}\sin^2\theta\,d\theta.$$

3.4.4 Integrating with Respect to θ

Recall that:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

Thus,

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\left[\int_0^{2\pi}1\,d\theta - \int_0^{2\pi}\cos(2\theta)\,d\theta\right].$$

Evaluate the integrals:

$$\int_0^{2\pi} 1 \, d\theta = 2\pi,$$

$$\int_0^{2\pi} \cos(2\theta) \, d\theta = 0.$$

Therefore,

$$\int_0^{2\pi} \sin^2 \theta \, d\theta = \frac{1}{2}(2\pi - 0) = \pi.$$

Substituting back:

$$\iiint_D y^2 \, dV = \frac{2}{5} \cdot \pi = \frac{2\pi}{5}.$$

3.5 Conclusion

The value of the triple integral is:

$$\iiint_D y^2 \, dV = \frac{2\pi}{5}.$$

4 Solution

We are tasked with evaluating the triple integral

$$\iiint_D y^2 \, dV,$$

where D is the region defined by:

$$x^2 + y^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad z^2 \leq 4x^2 + 4y^2.$$

4.1 Step 1: Understanding the Region D

4.1.1 Description of D

- **Base:** The inequality $x^2 + y^2 \leq 1$ represents a cylinder of radius 1 centered along the z-axis. - **Lower Bound:** $z \geq 0$ confines the region to above the xy-plane. - **Upper Bound:** $z^2 \leq 4x^2 + 4y^2$ can be rewritten as $z \leq 2\sqrt{x^2 + y^2}$, representing a double cone opening upwards and downwards. However, since $z \geq 0$, only the upper half-cone is relevant.

4.1.2 Visualization of D

The region D is a finite solid bounded below by the xy-plane and above by the cone $z = 2\sqrt{x^2 + y^2}$, within the cylinder $x^2 + y^2 \leq 1$.

4.2 Step 2: Converting to Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are defined by:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

where $r \geq 0$ and $\theta \in [0, 2\pi)$.

4.2.1 Expressing the Bounds in Cylindrical Coordinates

- **Cylinder $x^2 + y^2 \leq 1$:**

$$r^2 \leq 1 \implies r \leq 1.$$

- **Cone $z = 2\sqrt{x^2 + y^2}$:**

$$z = 2r.$$

- **Lower Bound $z = 0$:**

$$z \geq 0.$$

4.2.2 Region D in Cylindrical Coordinates

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 2r.$$

4.3 Step 3: Setting Up the Triple Integral

The triple integral in cylindrical coordinates is:

$$\iiint_D y^2 \, dV = \int_0^{2\pi} \int_0^1 \int_0^{2r} (r \sin \theta)^2 \cdot r \, dz \, dr \, d\theta.$$

Here, $y = r \sin \theta$ and $dV = r \, dz \, dr \, d\theta$.

4.4 Step 4: Evaluating the Integral

4.4.1 Simplifying the Integrand

$$y^2 = (r\sin\theta)^2 = r^2\sin^2\theta.$$

Thus, the integrand becomes:

$$y^2 \cdot r = r^2\sin^2\theta \cdot r = r^3\sin^2\theta.$$

The integral simplifies to:

$$\iiint_D y^2\,dV = \int_0^{2\pi}\int_0^1\int_0^{2r} r^3\sin^2\theta\,dz\,dr\,d\theta.$$

4.4.2 Integrating with Respect to z

$$\int_0^{2r} r^3\sin^2\theta\,dz = r^3\sin^2\theta\cdot[z]_0^{2r} = r^3\sin^2\theta\cdot 2r = 2r^4\sin^2\theta.$$

The integral becomes:

$$\iiint_D y^2\,dV = 2\int_0^{2\pi}\int_0^1 r^4\sin^2\theta\,dr\,d\theta.$$

4.4.3 Integrating with Respect to r

$$\int_0^1 r^4\,dr = \left[\frac{r^5}{5}\right]_0^1 = \frac{1}{5}.$$

Thus, the integral simplifies to:

$$\iiint_D y^2\,dV = 2\cdot\frac{1}{5}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{2}{5}\int_0^{2\pi}\sin^2\theta\,d\theta.$$

4.4.4 Integrating with Respect to θ

Recall that:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}.$$

Thus,

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\left[\int_0^{2\pi}1\,d\theta - \int_0^{2\pi}\cos(2\theta)\,d\theta\right].$$

Evaluate the integrals:

$$\int_0^{2\pi} 1\, d\theta = 2\pi,$$

$$\int_0^{2\pi} \cos(2\theta)\, d\theta = 0.$$

Therefore,

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \frac{1}{2}(2\pi - 0) = \pi.$$

Substituting back:

$$\iiint_D y^2\, dV = \frac{2}{5} \cdot \pi = \frac{2\pi}{5}.$$

4.5 Conclusion

The value of the triple integral is:

$$\iiint_D y^2\, dV = \frac{2\pi}{5}.$$