Notes for 18.02 Recitation 18

18.02 Recitation MW9

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Now, this one really upsets me. We have an out-of-tune player here ... Either you're deliberately out of tune and sabotaging my band, or you don't know you're out of tune, which, I'm afraid, is even worse.

— *Terence Fletcher in Whiplash*

This handout (and any other DLC's I write) are posted at `https://web.evanchen.cc/1802.html`.

§1 Triple integrals

You probably already saw me do some triple integrals in this class already. They're the same as double integrals with one extra integral, and so you shouldn't have to do much differently than before.

You can also do *change of variables* with three variables totally fine, in principle. But your Jacobian is 3×3, so evaluating it is annoying. Hence here are two examples whose Jacobian determinants you should just remember, because they come up often (like how we just remembered $\mathrm{d}A = \mathrm{d}r\,\mathrm{d}\theta$ in polar).

§1.1 Cylindrical coordinates

There's actually nothing new happening here — it's just polar coordinates with z tacked on.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

The volume scaling factor is unsurprisingly the same as the one for 2D polar coordinates, and you may have used it implicitly on some previous problem sets already:

❗ Memorize

$$\mathrm{d}V := \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

If you want to see this fully explicitly, you could compute the Jacobian

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

§1.2 Spherical coordinates (tomorrow)

I *hope* we match Poonen's conventions, in Figure 1. (Mathematicians and physicists use different notations.) The idea is that the projection of your point P onto the xy-plane will have polar coordinates

$(r\cos\theta, r\sin\theta)$. But then rather than using z to lift the point straight up, you rotate via some angle φ, getting a new distance ρ such that $r = \rho\sin\varphi$. (Greek letter names: rho, phi, theta. And φ, which is `\varphi` in LaTeX, can also be written as ϕ, which is `\phi` in LaTeX.)

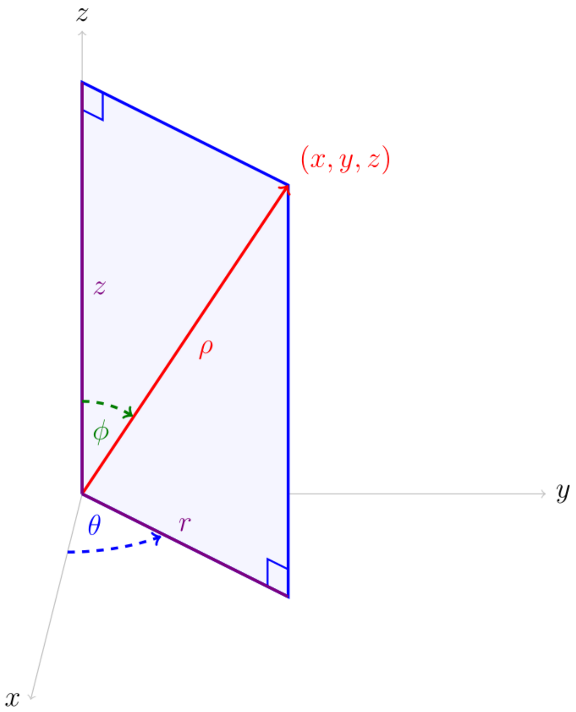

Figure 1: Spherical coordinates diagram from Poonen's notes.

Explicitly, the transition map $(\rho, \varphi, \theta) \mapsto (x, y, z)$ is defined by

$$\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi. \end{aligned}$$

To get the area scaling factor, we would compute the Jacobian $\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$. I'll do this on Wednesday, but spoiler, it turns out to be $\rho^2\sin\varphi$.

§2 Questions from official recitation

1. Let T be the region that is bounded by the surface $x = y^2$, and the planes $z = 0$ and $x + z = 1$. Sketch T and describe the projections of T onto the xy, xz and yz-plane. Using one of those projections (any one!), calculate the volume $\iiint_T \mathrm{d}V$.
2. Consider the integral $\iiint_R (2-2z)\,\mathrm{d}V$ where R is the pyramid with square base $-1 \le x \le 1$ and $-1 \le y \le 1$ in the xy-plane and vertex at $(0,0,1)$. Draw a picture of R and describe the cross sections of R for a fixed value of z. Use this to evaluate the integral by integrating with respect to x, then y, then z.
3. Using cylindrical coordinates, evaluate $\iiint_D y^2\,\mathrm{d}V$, where D is the region of points (x,y,z) such that $x^2+y^2 \le 1$, $z \ge 0$, and $z^2 \le 4x^2+4y^2$.
4. Evaluate the integral $\iiint_C \mathrm{d}V$ where C is the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$.