Quick answer key to Recitation 16

ChatGPT 4o

4 November 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Problem 1

Consider the vector field $\mathbf{F} = (y^2 + 2x)\mathbf{i} + axy\mathbf{j}$.

1.1 (a) Find the curl of F.

In two dimensions, the curl of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is given by:

$$\operatorname{curl}\mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Given:

$$P = y^2 + 2x \quad \text{and} \quad Q = axy$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = ay \quad \text{and} \quad \frac{\partial P}{\partial y} = 2y$$

Therefore, the curl is:

$$\operatorname{curl}\mathbf{F} = ay - 2y = (a-2)y$$

1.2 (b) For what values of a is $\mathbf{F}$ a conservative gradient field?

A vector field is conservative if its curl is zero throughout the domain. Setting the curl to zero:

$$(a-2)y = 0$$

For the curl to be zero for all y, we require:

$$a - 2 = 0 \implies a = 2$$

1.3 (c) For those values of a find a potential function.

For $a = 2$, the vector field becomes:

$$\mathbf{F} = (y^2 + 2x)\mathbf{i} + 2xy\mathbf{j}$$

We seek a function $f(x, y)$ such that:

$$\frac{\partial f}{\partial x} = y^2 + 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy$$

Integrate $\frac{\partial f}{\partial x}$ with respect to x:

$$f(x, y) = \int (y^2 + 2x)\, dx = y^2 x + x^2 + \phi(y)$$

Differentiate $f(x, y)$ with respect to y:

$$\frac{\partial f}{\partial y} = 2yx + \phi'(y)$$

Set equal to $\frac{\partial f}{\partial y}$:

$$2yx + \phi'(y) = 2xy \implies \phi'(y) = 0$$

Thus, $\phi(y)$ is a constant. The potential function is:

$$f(x, y) = xy^2 + x^2 + C$$

2 Problem 2

Consider the vector field $\mathbf{F} = e^{x+y}\left((x+a)\mathbf{i} + x\mathbf{j}\right)$.

2.1 (a) Find the curl of F.

Given:

$$P = e^{x+y}(x+a) \quad \text{and} \quad Q = e^{x+y}x$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = e^{x+y}x + e^{x+y} = e^{x+y}(x+1)$$

$$\frac{\partial P}{\partial y} = e^{x+y}(x+a)$$

Therefore, the curl is:

$$\operatorname{curl}\mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^{x+y}(x+1) - e^{x+y}(x+a) = e^{x+y}(1-a)$$

2.2 (b) For what values of a is $\mathbf{F}$ a conservative gradient field?

Set the curl to zero:

$$e^{x+y}(1-a)=0$$

Since $e^{x+y}>0$ for all x, y, we have:

$$1-a=0 \implies a=1$$

2.3 (c) For those values of a find a potential function.

For $a = 1$, the vector field simplifies to:

$$\mathbf{F} = e^{x+y}(x+1)\mathbf{i} + e^{x+y}x\mathbf{j}$$

We need a function $f(x, y)$ such that:

$$\frac{\partial f}{\partial x} = e^{x+y}(x+1) \quad \text{and} \quad \frac{\partial f}{\partial y} = e^{x+y}x$$

Integrate $\frac{\partial f}{\partial x}$ with respect to x:

$$f(x, y) = \int e^{x+y}(x+1)\,dx$$

Let $u = x+y$, then $du = dx$ (since y is treated as constant during integration with respect to x). Rewrite the integral:

$$f(x, y) = \int e^{u}(x+1)\,du$$

But since $x = u - y$, we have:

$$f(x, y) = \int e^{u}(u - y + 1)\,du$$

Simplify and integrate:

$$f(x, y) = \int e^{u}(u - y + 1)\,du = e^{u}(u - y)$$

Substitute back $u = x + y$:

$$f(x, y) = e^{x+y}(x + y - y) = e^{x+y}x$$

Compute the partial derivatives to verify:

$$\frac{\partial f}{\partial x} = e^{x+y}x + e^{x+y} = e^{x+y}(x+1)$$

$$\frac{\partial f}{\partial y} = e^{x+y}x$$

Thus, the potential function is:

$$f(x, y) = xe^{x+y}$$

3 Problem 3

Consider the vector field $\mathbf{G} = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$.

3.1 (a) Calculate the curl of G.

Given:

$$P = -\frac{y}{x^2 + y^2} \quad \text{and} \quad Q = \frac{x}{x^2 + y^2}$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial P}{\partial y} = -\left(\frac{(1)(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right) = -\left(\frac{x^2 - y^2}{(x^2 + y^2)^2}\right)$$

Compute the curl:

$$\operatorname{curl} \mathbf{G} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} - \left(-\frac{x^2 - y^2}{(x^2 + y^2)^2}\right) = 0$$

Therefore, the curl of **G** is zero everywhere except possibly at the origin.

3.2 (b) Show that G is not a gradient vector field by calculating the line integral $\int_C \mathbf{G} \cdot d\mathbf{r}$ for the closed curve C given by the unit circle, oriented counterclockwise.

Parametrize the unit circle:

$$x = \cos\theta, \quad y = \sin\theta, \quad \theta \in [0, 2\pi]$$

Compute differentials:

$$dx = -\sin\theta \, d\theta, \quad dy = \cos\theta \, d\theta$$

Evaluate $\mathbf{G}$ along C:

$$P = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{1} = -\sin\theta, \quad Q = \frac{x}{x^2+y^2} = \frac{\cos\theta}{1} = \cos\theta$$

Compute the dot product:

$$\mathbf{G} \cdot d\mathbf{r} = P\,dx + Q\,dy = (-\sin\theta)(-\sin\theta\,d\theta) + (\cos\theta)(\cos\theta\,d\theta) = (\sin^2\theta + \cos^2\theta)\,d\theta = d\theta$$

Integrate over C:

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_0^{2\pi} d\theta = 2\pi$$

Since the line integral around the closed curve C is non-zero, $\mathbf{G}$ is not a conservative vector field.