Notes for 18.02 Recitation 17

18.02 Recitation MW9

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6 November 2024

*Calvin, your test was an absolute disgrace! It's obvious you haven't read any of the material. Our first president was **not** Chef Boy-Ar-Dee and you ought to be ashamed to have turned in such preposterous answers!*

— *Miss Wormwood in Calvin and Hobbes, October 1, 1993*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

§1 Finding potential functions (see LAMV section 16)

How to find an anti-gradient with two variables

1. Let f denote the gradient function.
2. Integrate the given $\frac{\partial f}{\partial x}$ to get some equation of the form $f(x,y) = \text{expression} + g(y)$ for some function $g(y)$.
3. Put this into $\frac{\partial f}{\partial y}$ to find $g'(y)$.
4. Integrate get $g(y) = \text{expression} + C$ for some constant C.
5. Stitch everything together to output f.

Some examples were covered in class today, so here's one where we actually get stuck:

? Impossible question

Find f such that $\nabla f = \begin{pmatrix} 2y \\ x \end{pmatrix}$.

Solution. We start by integrating $\frac{\partial f}{\partial x}$ with respect to x to get $f(x,y) = \int 2y \,\mathrm{d}x = 2xy + g(y)$ to start. Then differentiating with respect to y gives $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2xy + g(y)) = 2x + g'(y)$. So then we set this equal and get $2x = g'(y) + x$ and so we need to solve $g'(y) = x$ which... wait! There's no way a function g can satisfy that for *every* x. What's going on? (It's tempting to write $g(y) = xy + C$, but that's a type error: These new function g can only depend on its arguments.)

In mathematics there's a concept of *proof by contradiction*: if you start from an assumption, and then do some logic and reasoning to reach an impossible conclusion, then the starting assumption was wrong. Here, the starting assumption that there was *some* function f such that $\nabla f = \begin{pmatrix} 2y \\ x \end{pmatrix}$. Starting from this assumption we found that there was a function $g : \mathbb{R} \to \mathbb{R}$ such that $g'(y) = x$ for every real number x, which is ridiculous. So our assumption was wrong: there can't be such function f. Not like 18.01 where "f exists but is hard to write down"; the function f literally cannot exist. □

Okay, so I bet you're all wondering now, "how can I tell if the question is impossible?". Well, one strategy would just be to run the recipe I showed you and see if it works out.

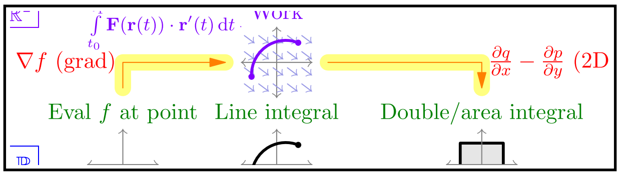
- If you find a function f that works, great.
- If you run into a contradiction, well, now you know it's impossible.

But that's a lot of work. We'd like a shortcut, and there is one.

> **Memorize: Criteria for 2D anti-gradient to exist**
>
> $\begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix}$ is conservative if and only if $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ (assume p and q are continuously differentiable).

One direction of this is the result $f_{xy} = f_{yx}$ you saw during the second derivative test. The nice thing is that it works the other way too: vector fields that pass this test will be conservative.

§2 Preview of Thu/Fri lectures (LAMV sections 29.5, 30.8, 31)



On Thursday, we'll introduce the **2D scalar curl**, the right highlighted Stokes arrow above.

- When you chain two red Stokes arrows in the poster, you always get 0. Indeed the 2D scalar is RHS − LHS of the criteria above.
- Every red Stokes arrow has a Stokes result; this time it will be **Green's theorem**.

On Friday you'll meet **2D flux** which I hate, but it is a rotated version of the work integral:

> **Definition of 2D flux**
>
> $$\text{2D flux} := \int_{t=\text{start time}}^{\text{stop time}} \mathbf{F}(\mathbf{r}(t)) \cdot (90^\circ \text{ clockwise rotation of } \mathbf{r}'(t))\,dt.$$

The "90° clockwise rotation of $\mathbf{r}'(t)$" is so awkward that you can bet people immediately made up a shorthand to sweep it under the rug. I think the usual notation is

$$\mathbf{n}\,ds := (90^\circ \text{ clockwise rotation of } \mathbf{r}'(t))\,dt$$

so that the above thing will usually be condensed to $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\,ds$. Because 2D flux is a rotated work integral, you'll get a Green's theorem for 2D flux too. Read LAMV 31.3 to see it.

§3 Recitation questions from the official course

1. Consider the vector field $\mathbf{F} = (y^2 + 2x)\mathbf{i} + axy\mathbf{j}$.
   - Find the curl of $\mathbf{F}$.
   - For what values of a is $\mathbf{F}$ a conservative gradient field?
   - For those values of a find a potential function.
2. Repeat Q1 for $\mathbf{F} = e^{x+y}((x+a)\mathbf{i} + x\mathbf{j})$.
3. Consider the vector field $\mathbf{G} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$. Calculate the curl of $\mathbf{G}$. Show that $\mathbf{G}$ is not a gradient vector field by calculating the line integral $\int_C G \cdot d\mathbf{r}$ for the closed curve C given by the unit circle, oriented counterclockwise.