Quick answer key to Recitation 16

ChatGPT 4o

4 November 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution

1.1 (a)

We are given the vector field $\mathbf{F}(x, y) = \langle x, -y \rangle$ on the xy-plane. We are to sketch this vector field and, on the same picture, draw the oriented path C from $(-1, 0)$ to $(0, -1)$ given by the unit circle in the quadrant where $x \leq 0$ and $y \leq 0$.

To sketch the vector field $\mathbf{F}(x, y) = \langle x, -y \rangle$:

- At any point (x, y), the vector points in the direction $\langle x, -y \rangle$. - For example: - At $(1, 0)$, $\mathbf{F} = \langle 1, 0 \rangle$, pointing right. - At $(0, 1)$, $\mathbf{F} = \langle 0, -1 \rangle$, pointing downward. - At $(-1, 0)$, $\mathbf{F} = \langle -1, 0 \rangle$, pointing left. - At $(0, -1)$, $\mathbf{F} = \langle 0, 1 \rangle$, pointing upward.

The path C is the quarter-circle from $(-1, 0)$ to $(0, -1)$ along the unit circle in the third quadrant ($x \leq 0$, $y \leq 0$). The path is oriented from $(-1, 0)$ to $(0, -1)$ in the clockwise direction.

1.2 (b)

Using the picture as a guide, we can estimate whether the line integral $\int_C x\,dx - y\,dy$ is positive, negative, or zero.

Along the path C:

- Both x and y are negative. - The vector field $\mathbf{F} = \langle x, -y \rangle$ evaluated along C has: - x-component negative (since $x < 0$). - y-component positive (since $-y > 0$ when $y < 0$).

The tangent vector to the path C is:

- Oriented clockwise from $(-1, 0)$ to $(0, -1)$. - At each point, the tangent vector points in the direction of motion along C.

Since the vector field $\mathbf{F}$ and the tangent vector to C are generally pointing in opposite directions (the x-components are both negative, but the y-components are opposite in sign), their dot product will be negative.

Therefore, we can expect that the line integral $\int_C x\,dx - y\,dy$ is **negative**.

1.3 Corrected Calculation for Part (c)

Re-parametrize C to match the orientation from $(-1, 0)$ to $(0, -1)$ (clockwise):

Let θ go from π to $\frac{3\pi}{2}$ **decreasing**:

$$\theta = \pi - t, \quad t \in [0, \tfrac{\pi}{2}]$$

Then:

$$x = \cos(\theta) = \cos(\pi - t) = -\cos t, \quad y = \sin(\theta) = \sin(\pi - t) = \sin t$$

Compute dx and dy:

$$dx = \sin t \, dt, \quad dy = \cos t \, dt$$

Compute the integral:

$$\int_C x \, dx - y \, dy = \int_{t=0}^{\frac{\pi}{2}} (-\cos t \cdot \sin t \, dt - \sin t \cdot \cos t \, dt) = -2 \int_0^{\frac{\pi}{2}} \cos t \sin t \, dt$$

Simplify using $\cos t \sin t = \frac{1}{2} \sin 2t$:

$$-2 \times \tfrac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2t \, dt = -\int_0^{\frac{\pi}{2}} \sin 2t \, dt$$

Integrate:

$$-\left(-\tfrac{1}{2} \cos 2t\right) \Big|_0^{\frac{\pi}{2}} = \tfrac{1}{2} (\cos \pi - \cos 0) = \tfrac{1}{2} (-1 - 1) = -1$$

Thus, the value of the integral is $\boxed{-1}$, consistent with the result from part (d).

1.4 (d)

We are to find a function $f(x, y)$ such that $\nabla f = \langle x, -y \rangle$.

Compute $f(x, y)$:

- Since $\dfrac{\partial f}{\partial x} = x$, integrate with respect to x:

$$f(x, y) = \int x\,dx + g(y) = \tfrac{1}{2}x^2 + g(y)$$

- Differentiate f with respect to y:

$$\frac{\partial f}{\partial y} = g'(y)$$

Given that $\dfrac{\partial f}{\partial y} = -y$, we have:

$$g'(y) = -y \implies g(y) = -\tfrac{1}{2}y^2 + C$$

Thus, the function is:

$$f(x, y) = \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + C$$

Using the fundamental theorem of line integrals:

$$\int_C x\,dx - y\,dy = f\left((0, -1)\right) - f\left((-1, 0)\right)$$

Compute $f(0, -1)$:

$$f(0, -1) = \tfrac{1}{2}(0)^2 - \tfrac{1}{2}(-1)^2 = -\tfrac{1}{2}$$

Compute $f(-1, 0)$:

$$f(-1, 0) = \tfrac{1}{2}(-1)^2 - \tfrac{1}{2}(0)^2 = \tfrac{1}{2}$$

Therefore:

$$\int_C x\,dx - y\,dy = f(0, -1) - f(-1, 0) = \left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right) = -1$$

2 Solution

2.1 (a)

We are asked to calculate the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = (x + y)\mathbf{i} + (xy)\mathbf{j}$, and C is the broken line running from $(0, 0)$ to $(2, 2)$ to $(0, 2)$.

2.1.1 Parameterization of the Path C

The path C consists of two segments:

1. Segment C_1: from $(0, 0)$ to $(2, 2)$.
2. Segment C_2: from $(2, 2)$ to $(0, 2)$.

Segment C_1: We can parametrize C_1 as:

$$\begin{aligned} x &= t, \\ y &= t, \\ t &\in [0, 2]. \end{aligned}$$

Compute $d\mathbf{r}$:

$$d\mathbf{r} = (dx, dy) = (dt, dt).$$

Compute $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} \cdot d\mathbf{r} = [(x + y), xy] \cdot (dx, dy) = (x + y)dx + xydy.$$

Since $x = y = t$, we have:

$$\begin{aligned} x + y &= t + t = 2t, \\ xy &= t \cdot t = t^2, \\ dx &= dt, \\ dy &= dt. \end{aligned}$$

Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = (2t)(dt) + (t^2)(dt) = [2t + t^2]dt.$$

Compute the integral over C_1:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{2} [2t + t^2]dt = \left[t^2 + \tfrac{1}{3}t^3\right]_0^2 = \left(4 + \tfrac{8}{3}\right) - 0 = \tfrac{12}{3} + \tfrac{8}{3} = \tfrac{20}{3}.$$

Segment C_2: We can parametrize C_2 as:

$$\begin{aligned} x &= 2 - t, \\ y &= 2, \\ t &\in [0, 2]. \end{aligned}$$

Compute $d\mathbf{r}$:

$$d\mathbf{r} = (dx, dy) = (-dt, 0).$$

Compute $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} \cdot d\mathbf{r} = [(x + y), xy] \cdot (dx, dy) = (x + y)dx + xydy.$$

Since $y = 2$, $x = 2 - t$, we have:

$$\begin{aligned} x + y &= (2 - t) + 2 = 4 - t, \\ xy &= (2 - t)(2) = 4 - 2t, \\ dx &= -dt, \\ dy &= 0. \end{aligned}$$

Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = (4 - t)(-dt) + (4 - 2t)(0) = -(4 - t)dt = (-4 + t)dt.$$

Compute the integral over C_2:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{2} (-4 + t)dt = \left[-4t + \tfrac{1}{2}t^2\right]_0^2 = (-8 + 2) - 0 = -6.$$

Total Integral: Add the integrals over C_1 and C_2:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \tfrac{20}{3} - 6 = \tfrac{20}{3} - \tfrac{18}{3} = \tfrac{2}{3}.$$

2.2 Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3}$$

2.3 (b)

We are asked to calculate the line integral:

$$\int_C (x^2 - y)\, dx + 2x\, dy$$

where C is the path from $(-1, 0)$ to $(1, 0)$ along the parabola $y = 1 - x^2$.

2.3.1 Parametrization of the Path C

We can parametrize C as:

$$x = t,$$
$$y = 1 - t^2,$$
$$t \in [-1, 1].$$

Compute dx and dy:

$$dx = dt, \quad dy = -2t\,dt.$$

Compute the integrand:

$$(x^2 - y)\,dx + 2x\,dy.$$

Substitute x, y, dx, and dy:

$$\begin{aligned} x^2 - y &= t^2 - (1 - t^2) = t^2 - 1 + t^2 = 2t^2 - 1, \\ dx &= dt, \\ 2x &= 2t, \\ dy &= -2t\,dt. \end{aligned}$$

Compute each term:

$$(x^2 - y)\,dx = (2t^2 - 1)dt, \quad 2x\,dy = 2t \cdot (-2t\,dt) = -4t^2 dt.$$

Add the terms:

$$(x^2 - y)\,dx + 2x\,dy = (2t^2 - 1)dt - 4t^2 dt = (-2t^2 - 1)dt.$$

Simplify:

$$(-2t^2 - 1)dt = -(2t^2 + 1)dt.$$

Compute the integral:

$$\int_{t=-1}^{1} -(2t^2 + 1)dt = -\left[\tfrac{2}{3}t^3 + t\right]_{-1}^{1} = -\left(\left(\tfrac{2}{3}(1)^3 + 1\right) - \left(\tfrac{2}{3}(-1)^3 + (-1)\right)\right).$$

Compute the values:

$$\begin{aligned} \text{At } t = 1: &\quad \tfrac{2}{3}(1) + 1 = \tfrac{2}{3} + 1 = \tfrac{5}{3}, \\ \text{At } t = -1: &\quad \tfrac{2}{3}(-1) + (-1) = -\tfrac{2}{3} - 1 = -\tfrac{5}{3}. \end{aligned}$$

Subtract:

$$\left(\tfrac{5}{3}\right) - \left(-\tfrac{5}{3}\right) = \tfrac{10}{3}.$$

Therefore,

$$\int_C (x^2 - y)\,dx + 2x\,dy = -\left(\tfrac{10}{3}\right) = -\tfrac{10}{3}.$$

2.4 Answer:

$$\int_C (x^2 - y)\,dx + 2x\,dy = -\frac{10}{3}$$

3 Solution

3.1 Part 1: Finding the Gradient Vector Field $\mathbf{F} = \nabla f(x, y)$

Given the function:

$$f(x, y) = \sin(x)\cos(y)$$

Compute the partial derivatives with respect to x and y:

$$\frac{\partial f}{\partial x} = \cos(x)\cos(y)$$
$$\frac{\partial f}{\partial y} = -\sin(x)\sin(y)$$

Therefore, the gradient vector field is:

$$\mathbf{F} = \nabla f(x, y) = (\cos(x)\cos(y),\ -\sin(x)\sin(y))$$

3.2 Part 2: Maximizing the Line Integral $\int_C \mathbf{F} \cdot d\mathbf{r}$

Since $\mathbf{F}$ is the gradient of f, the line integral over a path C from point A to point B is given by the fundamental theorem of line integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

To find the maximum possible value of the line integral as C ranges over all possible paths in the plane, we need to maximize the difference $f(B) - f(A)$.

3.2.1 Finding the Maximum and Minimum Values of $f(x, y)$

The function $f(x, y) = \sin(x)\cos(y)$ attains its maximum and minimum values based on the ranges of the sine and cosine functions:

$$\sin(x) \in [-1, 1]$$
$$\cos(y) \in [-1, 1]$$

Therefore, the maximum and minimum values of $f(x, y)$ are:

$$f_{\text{max}} = \sin(x_{\text{max}})\cos(y_{\text{max}}) = (1)(1) = 1$$
$$f_{\text{min}} = \sin(x_{\text{min}})\cos(y_{\text{min}}) = (-1)(-1) = 1$$

Wait, this suggests that both the maximum and minimum values are 1, which is incorrect. Let's reconsider.

Actually, the minimum value occurs when one of the functions is 1 and the other is -1:

$$f_{\text{min}} = \sin(x_{\text{min}})\cos(y_{\text{min}}) = (1)(-1) = -1 \quad \text{or} \quad (-1)(1) = -1$$

Therefore, the correct maximum and minimum values are:

$$f_{\text{max}} = 1$$
$$f_{\text{min}} = -1$$

3.2.2 Calculating the Maximum Value of the Line Integral

The maximum possible value of the line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) \leq f_{\text{max}} - f_{\text{min}} = 1 - (-1) = 2$$

Thus, the maximum possible value of the line integral is $\boxed{2}$.

3.3 Conclusion

The gradient vector field is:

$$\mathbf{F} = \nabla f(x, y) = (\cos(x)\cos(y),\ -\sin(x)\sin(y))$$

The maximum possible value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as C ranges over all possible paths in the plane is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} \leq 2$$

Therefore, the maximum value is $\boxed{2}$.