Notes for 18.02 Recitation 16

18.02 Recitation MW9

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You don't get to choose who Riley is. Anxiety, you need to let her go.

— *Joy to Anxiety in Inside Out 2*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

§1 Announcements

- My LAMV notes should be basically complete up to midterm 3 now.
  - I encourage you to read Part Golf and Part Hotel up to 2D flux.
  - Section 16, "anti-gradient", is also relevant. It was retroactively added to Part Echo.
- I will have **extra office hours** today at
  - 3:30pm-4:00pm in 2-131
  - 4:00pm-5:00pm in 2-135.
- The **mock midterm 3** is scheduled for **Wed Nov 13, 6pm-8pm, in 1-190**. Email me if you want to try the problems early.

§2 Shorthands introduced thus far

In $\mathbb{R}^2$, suppose $\mathbf{F}(x,y) = \binom{p(x,y)}{q(x,y)}$. Let path $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$ trace out a curve $\mathcal{C}$. All of the following notations mean the same thing:

$$\int_{t=\text{start time}}^{\text{stop time}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} (p\,\mathrm{d}x + q\,\mathrm{d}y).$$

These are called the **work** of $\mathbf{F}$ on $\mathcal{C}$, aka the **line integral**. The work is a scalar quantity (there is a dot product inside the integrand, so it returns a number).

§3 Computing line integrals in general

Recipe for computing line integrals in general

Suppose we want to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

1. First, check if the vector field is conservative by seeing if the curl is zero.
   - If so, don't bother parametrizing $\mathcal{C}$. Find a potential function f for the vector field $\mathbf{F}$ and use the FTC as a shortcut.
2. **Not yet covered in class**: If $\mathcal{C}$ is a closed loop in $\mathbb{R}^2$, use Green's theorem instead.
3. If both of these fail, fall back the parametrization recipe.

> **❗ Memorize: FTC for line integrals**
>
> Suppose $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a conservative vector field, given by $\mathbf{F} = \nabla f$ for some potential function $f : \mathbb{R}^n \to \mathbb{R}$. Then for any curve $\mathcal{C}$ from a point P to a point Q we have
>
> $$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P).$$

§4 Massive spoilers

I'm just gonna spoil the answers to all three of Maulik's questions from his notes:

1. **Are all conservative fields gradients?**

   Answer: Heck yes! One fewer word to remember 🎉

2. **Given $\mathbf{F}$, how can we tell if it's a gradient?**

   > **❗ Memorize: Conservative $\iff$ 2D scalar curl is zero**
   >
   > Assume here the vector field is continuously differentiable and defined everywhere on $\mathbb{R}^2$. A vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(x,y) = \binom{p(x,y)}{q(x,y)}$ is conservative if and only if
   >
   > $$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

   When $\mathbf{F} = \nabla f$, that condition is just saying $f_{xy} = f_{yx}$, which is something you already saw back when doing the second derivative test. If you like ∂ notation better, this could also be written as

   $$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}.$$

3. **If it is, how can we find a potential function?**

   Answer: read section 16 of LAMV for the recipe.

§5 Recitation questions from official course

1a Sketch the vector field $\langle x, -y\rangle$ on the xy-plane. On the same picture, draw the oriented path C from $(-1,0)$ to $(0,-1)$ given by the unit circle in the quadrant $x \le 0, y \le 0$.

1b Using the picture as a guide, is $\int_C x\,dx - y\,dy$ positive, negative, or zero?

1c Using a parametrization of C, calculate the line integral $\int_C x\,dx - y\,dy$ exactly.

1d Find a function $f(x,y)$ whose gradient is $\langle x, -y\rangle$, and use the fundamental theorem of calculus to calculate $\int_C x\,dx - y\,dy$ in another way.

2a Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is the vector field $(x+y)\mathbf{i} + xy\mathbf{j}$ and C is the broken line running from $(0,0)$ to $(2,2)$ to $(0,2)$.

2b Calculate the line integral $\int_C (x^2 - y)\,dx + 2x\,dy$ where C is the path from $(-1,0)$ to $(1,0)$ along the parabola $y = 1 - x^2$.

3 Let $f(x,y) = \sin(x)\cos(y)$. Find the gradient vector field $\mathbf{F} = \nabla f(x,y)$. What is the maximum possible value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ as C ranges over all possible paths in the plane?