Quick answer key to Recitation 15

ChatGPT 4o

Wednesday, 30 October 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

We define the parallelogram coordinates (u, v) by the transformations $x = u + v$ and $y = u - v$.

1 Problem Solution

1. **Sketch the unit square in (u, v)-coordinates on the xy-plane.**

   To map the unit square with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$ in (u, v)-coordinates onto the xy-plane, we calculate the coordinates of each vertex under the transformations $x = u + v$ and $y = u - v$:

   - $(u, v) = (0, 0) : \quad (x, y) = (0 + 0, 0 - 0) = (0, 0).$
   - $(u, v) = (1, 0) : \quad (x, y) = (1 + 0, 1 - 0) = (1, 1).$
   - $(u, v) = (0, 1) : \quad (x, y) = (0 + 1, 0 - 1) = (1, -1).$
   - $(u, v) = (1, 1) : \quad (x, y) = (1 + 1, 1 - 1) = (2, 0).$

   Therefore, the unit square in (u, v)-coordinates maps to a parallelogram in (x, y)-coordinates with vertices at $(0, 0), (1, 1), (1, -1)$, and $(2, 0)$.

2. **Evaluate $dxdy$ in terms of $dudv$.**

   To find $dx\,dy$ in terms of $du\,dv$, we compute the Jacobian determinant:

   $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

   Since $x = u + v$ and $y = u - v$:

   $$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1.$$

   Thus,

   $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -2.$$

   Therefore,

   $$dx\,dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv = 2\,du\,dv.$$

3. **Evaluate the integrals $I_1 = \iint_P dA$ and $I_2 = \iint_P (x^2 - y^2)\,dA$ using the change of variables.**

Integral I_1

Since P is the region in xy-coordinates with vertices $(0,0),(1,1),(1,-1),(2,0)$, it corresponds to the unit square $0 \leq u \leq 1$, $0 \leq v \leq 1$ in (u,v)-coordinates. Thus,

$$I_1 = \iint_P dA = \int_0^1 \int_0^1 2\,du\,dv = 2\int_0^1 \int_0^1 du\,dv.$$

Evaluating the inner integral with respect to u:

$$\int_0^1 \int_0^1 du\,dv = \int_0^1 [u]_0^1\,dv = \int_0^1 1\,dv = [v]_0^1 = 1.$$

So,

$$I_1 = 2 \cdot 1 = 2.$$

Integral I_2

To evaluate $I_2 = \iint_P (x^2 - y^2)\,dA$, we express $x^2 - y^2$ in terms of u and v. Since $x = u + v$ and $y = u - v$,

$$x^2 - y^2 = (u+v)^2 - (u-v)^2.$$

Expanding each term:

$$(u+v)^2 = u^2 + 2uv + v^2 \quad \text{and} \quad (u-v)^2 = u^2 - 2uv + v^2.$$

Therefore,

$$x^2 - y^2 = (u^2 + 2uv + v^2) - (u^2 - 2uv + v^2) = 4uv.$$

So,

$$I_2 = \iint_P (x^2 - y^2)\,dA = \int_0^1 \int_0^1 4uv \cdot 2\,du\,dv = 8\int_0^1 \int_0^1 uv\,du\,dv.$$

Now, evaluate the inner integral with respect to u:

$$\int_0^1 \int_0^1 uv\,du\,dv = \int_0^1 \left[\frac{u^2}{2}\right]_0^1 v\,dv = \int_0^1 \frac{1}{2} v\,dv = \frac{1}{2}\int_0^1 v\,dv = \frac{1}{2}\left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Therefore,

$$I_2 = 8 \cdot \frac{1}{4} = 2.$$

1.1 Conclusion

The values of the integrals are:

$$I_1 = 2, \quad I_2 = 2.$$

2 Solution

We are asked to evaluate the integral:

$$I = \iint_R x^2 y^4 \, dA$$

where R is the region bounded by $xy = 4$, $xy = 8$, $y = x$, and $y = 4x$. We will use the transformation:

$$x = 2\sqrt{\frac{u}{v}}, \quad y = 2\sqrt{uv}$$

2.1 Step 1: Change of Variables

First, we express u and v in terms of x and y to find the limits of integration and compute the Jacobian determinant.

From the given transformation:

$$xy = \left(2\sqrt{\frac{u}{v}}\right)\left(2\sqrt{uv}\right) = 4u$$

Thus,

$$u = \frac{xy}{4}$$

Also,

$$v = \frac{y}{x}$$

2.2 Step 2: Determining the Region S in the uv-Plane

The region R in the xy-plane is bounded by:

1. $xy = 4 \implies u = \frac{4}{4} = 1$
2. $xy = 8 \implies u = \frac{8}{4} = 2$
3. $y = x \implies v = \frac{x}{x} = 1$
4. $y = 4x \implies v = \frac{4x}{x} = 4$

Therefore, in the uv-plane, the region S is defined by $1 \leq u \leq 2$ and $1 \leq v \leq 4$.

2.3 Step 3: Computing the Jacobian Determinant

We compute the Jacobian determinant $J(u,v) = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

First, compute the partial derivatives:

$$\frac{\partial x}{\partial u} = 2\left(\frac{u}{v}\right)^{-1/2} \cdot \frac{1}{2}v^{-1} = \frac{1}{u^{1/2}v^{1/2}}$$
$$\frac{\partial x}{\partial v} = 2\left(\frac{u}{v}\right)^{-1/2} \cdot \left(-\frac{u}{2v^2}\right) = -\frac{u^{1/2}}{v^{3/2}}$$
$$\frac{\partial y}{\partial u} = 2(uv)^{-1/2} \cdot \frac{1}{2}v = \frac{v^{1/2}}{u^{1/2}}$$
$$\frac{\partial y}{\partial v} = 2(uv)^{-1/2} \cdot \frac{1}{2}u = \frac{u^{1/2}}{v^{1/2}}$$

Compute the determinant:

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \dfrac{1}{u^{1/2}v^{1/2}} & -\dfrac{u^{1/2}}{v^{3/2}} \\ \dfrac{v^{1/2}}{u^{1/2}} & \dfrac{u^{1/2}}{v^{1/2}} \end{vmatrix} \\ &= \left(\frac{1}{u^{1/2}v^{1/2}} \cdot \frac{u^{1/2}}{v^{1/2}}\right) - \left(-\frac{u^{1/2}}{v^{3/2}} \cdot \frac{v^{1/2}}{u^{1/2}}\right) \\ &= \frac{1}{v} + \frac{1}{v} = \frac{2}{v} \end{aligned}$$

2.4 Step 4: Transforming the Integral

We substitute x, y, and dA into the integral:

$$I = \iint_S x^2 y^4 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Compute x^2 and y^4:

$$x^2 = \left(2\sqrt{\frac{u}{v}}\right)^2 = 4\left(\frac{u}{v}\right)$$
$$y^4 = \left(2\sqrt{uv}\right)^4 = 16(uv)^2 = 16u^2v^2$$

Thus,

$$x^2y^4 = 4\left(\frac{u}{v}\right) \times 16u^2v^2 = 64u^3v$$

Including the Jacobian determinant:

$$x^2y^4 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = 64u^3v \times \frac{2}{v} = 128u^3$$

2.5 Step 5: Evaluating the Integral

The integral becomes:

$$I = \int_{u=1}^{2} \int_{v=1}^{4} 128u^3 \, dv \, du$$

Since the integrand is independent of v, we can integrate with respect to v:

$$\int_{v=1}^{4} 128u^3 \, dv = 128u^3(4-1) = 384u^3$$

Now, integrate with respect to u:

$$I = \int_{u=1}^{2} 384u^3 \, du = 384 \left[\frac{u^4}{4} \right]_1^2 = 384 \left(\frac{16}{4} - \frac{1}{4} \right) = 384 \left(\frac{15}{4} \right) = 96 \times 15 = 1440$$

2.6 Conclusion

Therefore, the value of the integral is:

$$I = 1440$$

3 Solution

We are asked to find the area of the region in the plane defined by the inequalities:

$$\begin{cases} 1 \le xy \le 3, \\ 2 \le xy^2 \le 10, \\ x \ge 0. \end{cases}$$

3.1 Step 1: Choose an Appropriate Change of Variables

We notice that the inequalities involve xy and xy^2. Let's define new variables to simplify the region:

$$u = xy, \quad v = y.$$

This substitution will help us transform the given region into a simpler one in the uv-plane.

3.2 Step 2: Express x and y in Terms of u and v

From the substitution:

$$x = \frac{u}{y} = \frac{u}{v}.$$

3.3 Step 3: Compute the Jacobian Determinant

We need to compute the Jacobian determinant $J(u, v)$ of the transformation:

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|.$$

Compute the partial derivatives:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{1}{v}, & \frac{\partial x}{\partial v} &= -\frac{u}{v^2}, \\ \frac{\partial y}{\partial u} &= 0, & \frac{\partial y}{\partial v} &= 1. \end{aligned}$$

Compute the determinant:

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}.$$

3.4 Step 4: Transform the Inequalities to the uv-Plane

Transform the inequalities using the substitution:

- $1 \leq xy = u \leq 3$.
- $2 \leq xy^2 = uv \leq 10$.
- $x \geq 0 \implies \frac{u}{v} \geq 0$.

Since $v = y$ and $x = \frac{u}{v} \geq 0$, we have:

$$\frac{u}{v} \geq 0 \implies u \text{ and } v \text{ have the same sign.}$$

Given $1 \leq u \leq 3$ (so $u > 0$), it follows that $v > 0$.

Thus, the inequalities become:

$$\begin{cases} 1 \leq u \leq 3, \\ 2 \leq uv \leq 10, \\ v > 0. \end{cases}$$

3.5 Step 5: Determine the Limits of Integration

For u between 1 and 3, v varies according to:

$$2 \leq uv \leq 10 \implies \frac{2}{u} \leq v \leq \frac{10}{u}.$$

Therefore, the limits are:

$$u \in [1, 3],$$
$$v \in \left[\frac{2}{u}, \frac{10}{u}\right].$$

3.6 Step 6: Set Up the Integral

The area A is given by:

$$A = \iint_{\text{Region}} dx\, dy = \iint_{\text{Region}} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv = \iint_{\text{Region}} \frac{1}{v}\, du\, dv.$$

3.7 Step 7: Evaluate the Integral

Compute the integral:

$$A = \int_{u=1}^{3} \int_{v=\frac{2}{u}}^{\frac{10}{u}} \frac{1}{v} \, dv \, du.$$

First, integrate with respect to v:

$$\int_{\frac{2}{u}}^{\frac{10}{u}} \frac{1}{v} \, dv = \ln\left(\frac{10}{u}\right) - \ln\left(\frac{2}{u}\right) = \ln\left(\frac{10}{u} \times \frac{u}{2}\right) = \ln\left(\frac{10}{2}\right) = \ln 5.$$

So the integral simplifies to:

$$A = \ln 5 \int_{u=1}^{3} du = \ln 5 \times (3 - 1) = 2 \ln 5.$$

3.8 Conclusion

The area of the region is:

$$A = 2\ln 5.$$