Notes for 18.02 Recitation 15

18.02 Recitation MW9

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Vim is the greatest editor since the stone chisel.

— *Jose Unpingco, in `doc/vim/quotes.txt`*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

§1 Example from LAMV 23

Suppose you want to integrate over the region in the top half of the figure below. Then we can define a transition map $\mathbf{T}$ so that $u = \frac{y}{x}$ ranges from $\frac{1}{4}$ to 4 while $v = xy$ ranges from $\frac{16}{25}$ to $\frac{16}{9}$, giving the bottom half of the figure.

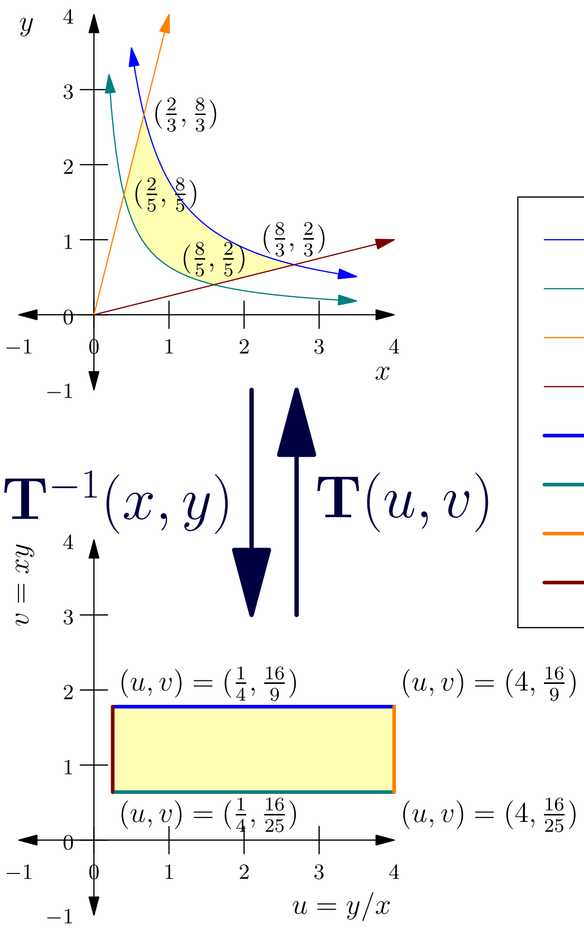

Figure 1: We use a (u, v) rectangle as a transition map to do cartography on the region $\mathcal{R}$.

Memorize: Change of variables

Suppose you need to integrate $\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ and you have a transition map $\mathbf{T}(u,v) : \mathcal{S} \to \mathcal{R}$. Then the transition map lets you change the integral as follows:

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{S}} f(u,v) |\det J_{\mathbf{T}}|\,\mathrm{d}u\,\mathrm{d}v$$

Alternatively, if it's easier to compute $J_{\mathbf{T}^{-1}}$, the following formula also works:

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{S}} \frac{f(u,v)}{|\det J_{\mathbf{T}^{-1}}|}\,\mathrm{d}u\,\mathrm{d}v$$

Here $|\det J_{\mathbf{T}}|$ is called the **area scaling factor**: it's the absolute value of the determinant of the Jacobian matrix. It's indeed true that

$$\det J_{\mathbf{T}^{-1}} = \frac{1}{\det(J_{\mathbf{T}})}$$

which means that if your transition map has a nicer inverse than the original, you might prefer to use that instead.

§2 Recitation questions from official course

Today's guest instructor is Lichen.

1. Define the parallelogram coordinates (u,v) by $x = u + v$ and $y = u - v$.
   1. Sketch what the unit square with coordinates $(0,0)$; $(1,0)$; $(0,1)$; $(1,1)$ in (u,v) coordinates looks like on the xy-plane.
   2. Evaluate $\mathrm{d}x\,\mathrm{d}y$ in terms of $\mathrm{d}u\,\mathrm{d}v$.
   3. Define the region P as the parallelogram with vertices at $(0,0)$, $(1,1)$, $(1,-1)$, and $(2,0)$ in xy-coordinates. Evaluate the double integrals

   $$I_1 = \iint_P \mathrm{d}A \quad I_2 = \iint_P (x^2 - y^2)\,\mathrm{d}A$$

   by performing the change of variables to parallelogram coordinates.

2. Evaluate the following integral using the given change of coordinates:

$$\iint_R x^2 y^4\,\mathrm{d}A$$

where R is the region in the first quadrant bounded by $xy = 4$; $xy = 8$; $y = x$; and $y = 4x$ using the transformation $x = 2\sqrt{u/v}$ and $y = 2\sqrt{uv}$.

3. Find the area of the region of points (x,y) in the plane satisfying the inequalities $1 \le xy \le 3$, $2 \le xy^2 \le 10$, and $x \ge 0$ using an appropriate change of variables.