Quick answer key to Recitation 14

ChatGPT 4o

Monday, 28 October 2024

Use the table of contents below to skip to a specific part without seeing spoilers to the other parts.

I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution 1

We are given the iterated integral:

$$\int_0^2 \int_{y/2}^1 e^{-x^2}\,dx\,dy.$$

To simplify the evaluation, we will swap the order of integration.

1.1 Step 1: Determine the region of integration

The given bounds indicate that y ranges from 0 to 2 and, for each fixed y, x ranges from $y/2$ to 1. This region can be described by the inequalities:

$$0 \leq y \leq 2 \quad \text{and} \quad \frac{y}{2} \leq x \leq 1.$$

To reverse the order of integration, we express y in terms of x.

1. From $x \geq y/2$, we get $y \leq 2x$. 2. Since $y \leq 2$, we have $x \leq 1$.

Thus, the region can be described by $0 \leq x \leq 1$ and $0 \leq y \leq 2x$.

1.2 Step 2: Rewrite the integral with the new bounds

Swapping the order of integration, the integral becomes:

$$\int_0^1 \int_0^{2x} e^{-x^2} \, dy \, dx.$$

1.3 Step 3: Evaluate the inner integral

Since e^{-x^2} is independent of y, we can factor it out:

$$\int_0^1 \int_0^{2x} e^{-x^2}\, dy\, dx = \int_0^1 e^{-x^2} \left(\int_0^{2x} 1\, dy \right) dx.$$

Now evaluate the inner integral with respect to y:

$$\int_0^{2x} 1\, dy = [y]_0^{2x} = 2x.$$

Substituting back, we have:

$$\int_0^1 \int_0^{2x} e^{-x^2}\, dy\, dx = \int_0^1 2xe^{-x^2}\, dx.$$

1.4 Step 4: Evaluate the outer integral

Now we integrate with respect to x:

$$\int_0^1 2xe^{-x^2}\,dx.$$

Let $u = x^2$, so $du = 2x\,dx$. When $x = 0$, $u = 0$; and when $x = 1$, $u = 1$. The integral becomes:

$$\int_0^1 e^{-x^2} \cdot 2x\,dx = \int_0^1 e^{-u}\,du.$$

Now integrate with respect to u:

$$\int_0^1 e^{-u}\,du = [-e^{-u}]_0^1 = -e^{-1} + e^0 = 1 - \frac{1}{e}.$$

1.5 Conclusion

The value of the integral is:

$$\int_0^2 \int_{y/2}^1 e^{-x^2}\, dx\, dy = 1 - \frac{1}{e} \approx 0.6321.$$

2 Solution 2

We are tasked with evaluating the integral:

$$\int\int_D \frac{dA}{3 + x^2 + y^2},$$

where D is the region defined by $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 9$. This region corresponds to the first quadrant of a disk of radius 3 centered at the origin.

2.1 Step 1: Convert to polar coordinates

In polar coordinates, we have:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad dA = r\,dr\,d\theta.$$

The expression $x^2 + y^2$ becomes r^2, so the integrand simplifies to:

$$\frac{1}{3 + x^2 + y^2} = \frac{1}{3 + r^2}.$$

2.2 Step 2: Set up the bounds in polar coordinates

The region D corresponds to $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 9$. In polar coordinates: - r ranges from 0 to 3 (since $x^2 + y^2 \leq 9$ implies $r \leq 3$). - θ ranges from 0 to $\frac{\pi}{2}$ (covering the first quadrant).

Thus, the integral becomes:

$$\int\int_D \frac{dA}{3 + x^2 + y^2} = \int_0^{\pi/2} \int_0^3 \frac{r}{3 + r^2}\, dr\, d\theta.$$

2.3 Step 3: Evaluate the inner integral with respect to r

We first evaluate the inner integral:

$$\int_0^3 \frac{r}{3+r^2}\,dr.$$

To integrate this, we use the substitution $u = 3 + r^2$, so $du = 2r\,dr$ or $\frac{du}{2} = r\,dr$. When $r = 0$, $u = 3$; and when $r = 3$, $u = 12$. Thus, the integral becomes:

$$\int_0^3 \frac{r}{3+r^2}\,dr = \int_3^{12} \frac{1}{u} \cdot \frac{du}{2} = \frac{1}{2}\int_3^{12} \frac{1}{u}\,du.$$

Now integrate with respect to u:

$$\frac{1}{2}\int_3^{12} \frac{1}{u}\,du = \frac{1}{2}\left[\ln|u|\right]_3^{12} = \frac{1}{2}\left(\ln(12) - \ln(3)\right).$$

Using the property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$, we get:

$$\frac{1}{2}\left(\ln(12) - \ln(3)\right) = \frac{1}{2}\ln\left(\frac{12}{3}\right) = \frac{1}{2}\ln(4) = \frac{1}{2} \cdot 2\ln(2) = \ln(2).$$

So, the value of the inner integral is $\ln(2)$.

2.4 Step 4: Evaluate the outer integral with respect to θ

The outer integral is:

$$\int_0^{\pi/2} \ln(2)\, d\theta = \ln(2) \int_0^{\pi/2} 1\, d\theta = \ln(2) \cdot \frac{\pi}{2} = \frac{\pi}{2} \ln(2).$$

2.5 Conclusion

The value of the integral is:

$$\int\int_D \frac{dA}{3 + x^2 + y^2} = \frac{\pi}{2}\ln(2).$$

3 Solution 3 (took like six tries for GPT to get this right and a lot of nudging)

We are given the density function $\delta(x, y) = \sqrt{x^2 + y^2}$ and the region defined by

$$x^2 + (y - 1)^2 \leq 1.$$

This region is a disk of radius 1 centered at $(0, 1)$ in the xy-plane. We want to find the mass of the shape, which is given by:

$$M = \iint_D \delta(x, y)\, dA = \iint_D \sqrt{x^2 + y^2}\, dA.$$

3.1 Step 1: Convert to polar coordinates centered at the origin

In polar coordinates centered at $(0, 0)$, we have:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and the differential area element $dA = r\,dr\,d\theta$. The density function simplifies to:

$$\delta(x, y) = \sqrt{x^2 + y^2} = r.$$

3.2 Step 2: Determine the region D in polar coordinates

The region D is a disk of radius 1 centered at $(0, 1)$, which corresponds to all points (x, y) such that $x^2 + (y - 1)^2 \leq 1$.

In polar coordinates, this region D can be described by:

$$0 \leq r \leq 2\sin\theta, \quad 0 \leq \theta \leq \pi.$$

This is because $r = 2\sin\theta$ describes a circle of radius 1 centered at $(0, 1)$, and θ only goes from 0 to π to capture the upper half-plane where $y \geq 0$.

3.3 Step 3: Set up the integral for the mass

The mass M is given by:

$$M = \int_0^{\pi} \int_0^{2\sin\theta} r \cdot r \, dr \, d\theta = \int_0^{\pi} \int_0^{2\sin\theta} r^2 \, dr \, d\theta.$$

3.4 Step 4: Evaluate the inner integral with respect to r

We first integrate with respect to r:

$$\int_0^{2\sin\theta} r^2\, dr = \left[\frac{r^3}{3}\right]_0^{2\sin\theta} = \frac{(2\sin\theta)^3}{3} = \frac{8\sin^3\theta}{3}.$$

3.5 Step 5: Evaluate the outer integral with respect to θ

Now, we integrate with respect to θ:

$$M = \int_0^{\pi} \frac{8\sin^3\theta}{3}\, d\theta = \frac{8}{3}\int_0^{\pi} \sin^3\theta\, d\theta.$$

To evaluate $\int \sin^3\theta\, d\theta$, we use the identity $\sin^3\theta = \sin\theta \cdot (1 - \cos^2\theta)$ and let $u = \cos\theta$, $du = -\sin\theta\, d\theta$:

$$\int \sin^3\theta\, d\theta = \int (1 - u^2)(-du) = \int (u^2 - 1)\, du = \left[\frac{u^3}{3} - u\right]_1^{-1}.$$

Evaluating this from $\theta = 0$ to π, we get:

$$\int_0^{\pi} \sin^3\theta\, d\theta = \frac{4}{3}.$$

Thus, the mass M is:

$$M = \frac{8}{3} \cdot \frac{4}{3} = \frac{32}{9}.$$

3.6 Conclusion

The mass of the shape is:

$$\boxed{\frac{32}{9}}.$$

4 Solution 4

We are tasked with evaluating the integral:

$$\int\int_D \frac{dA}{\sqrt{1 - x^2 - y^2}},$$

where D is the region defined by $x \geq 0$ and $x^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$. This region is a semicircular disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$ in the first quadrant.

4.1 Step 1: Interpret the region D

The inequality $x^2 + (y - \frac{1}{2})^2 \leq \frac{1}{4}$ describes a disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$. Since $x \geq 0$, we are only considering the right half of this disk, which lies in the first quadrant.

4.2 Step 2: Convert to polar coordinates

To simplify the integral, we use polar coordinates centered at the origin. In polar coordinates, we have:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and the differential area element $dA = r\,dr\,d\theta$.

In these coordinates, the integrand $\frac{1}{\sqrt{1-x^2-y^2}}$ becomes:

$$\frac{1}{\sqrt{1-x^2-y^2}} = \frac{1}{\sqrt{1-r^2}}.$$

4.3 Step 3: Determine the limits for r and θ

Since the region D is a semicircular disk of radius $\frac{1}{2}$ in the first quadrant, we have: - r ranges from 0 to $\frac{1}{2}$, - θ ranges from 0 to π.

4.4 Step 4: Set up the integral

Substituting into the integral, we get:

$$\int\int_D \frac{dA}{\sqrt{1 - x^2 - y^2}} = \int_0^{\pi} \int_0^{1/2} \frac{r}{\sqrt{1 - r^2}} \, dr \, d\theta.$$

4.5 Step 5: Evaluate the inner integral with respect to r

We first evaluate the inner integral:

$$\int_0^{1/2} \frac{r}{\sqrt{1-r^2}}\,dr.$$

To integrate this, we use the substitution $u = 1 - r^2$, so $du = -2r\,dr$ or $\frac{-du}{2} = r\,dr$. When $r = 0$, $u = 1$; and when $r = \frac{1}{2}$, $u = \frac{3}{4}$. Thus, the integral becomes:

$$\int_0^{1/2} \frac{r}{\sqrt{1-r^2}}\,dr = \int_1^{3/4} \frac{-1}{2\sqrt{u}}\,du = -\frac{1}{2}\int_1^{3/4} u^{-1/2}\,du.$$

Now integrate with respect to u:

$$-\frac{1}{2}\int_1^{3/4} u^{-1/2}\,du = -\frac{1}{2}\left[2\sqrt{u}\right]_1^{3/4} = -\sqrt{u}\Big|_1^{3/4}.$$

Evaluating this, we get:

$$-\sqrt{\frac{3}{4}} + \sqrt{1} = -\frac{\sqrt{3}}{2} + 1 = 1 - \frac{\sqrt{3}}{2}.$$

4.6 Step 6: Evaluate the outer integral with respect to θ

The outer integral is:

$$\int_0^\pi \left(1 - \frac{\sqrt{3}}{2}\right) d\theta = \left(1 - \frac{\sqrt{3}}{2}\right) \int_0^\pi 1\, d\theta = \left(1 - \frac{\sqrt{3}}{2}\right) \cdot \pi.$$

4.7 Conclusion

The value of the integral is:

$$\int\int_D \frac{dA}{\sqrt{1 - x^2 - y^2}} = \pi \left(1 - \frac{\sqrt{3}}{2}\right).$$