Notes for 18.02 Recitation 14

18.02 Recitation MW9

Evan Chen

26 October 2024

The real fun of living wisely is that you get to be smug about it.

— *Hobbes from Calvin and Hobbes*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

- You can vote on times for review session 3 at https://forms.gle/Xh8B5C2tPVtD5xDU6.
  - I'm amazed so many people check their email minute-ly on a Sunday morning.
- LAMV had a wrong version of change of formulas for a week-ish. Fair warning.
- Wednesday's recitation will have Lichen as a guest instructor. (I'll be in the audience.)

§1 From last time

If you missed Wednesday's recitation, some advice:

- For this class, take $\mathrm{d}A$ as a shorthand for $\mathrm{d}x\,\mathrm{d}y$. (See LAMV 25.1.)
- The result $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ is a special case of *change of variables*, covered in tomorrow's lecture and recitation on Wednesday (led by Lichen).
- Write regions as inequalities, e.g. if you have the region cut out by $y = x^2$ and $y = x + 2$, I would prefer to write $y \geq x^2$ (above parabola) and $y \leq x + 2$ (below line).
- I suggest writing the variables into the integration signs, e.g., rather than $\int_0^2 \int_{\frac{y}{2}}^1 e^{-x^2}\,\mathrm{d}x\,\mathrm{d}y$ I think this is clearer as $\int_{y=0}^2 \int_{x=\frac{y}{2}}^1 e^{-x^2}\,\mathrm{d}x\,\mathrm{d}y$. This prevents you from having to read the $\mathrm{d}x\,\mathrm{d}y$ at the end backwards, and lets you get away with just writing $\mathrm{d}A$ for shorthand.

§2 Polar coordinates is a special case of change of variables (tmrw)

The map

$$\mathbf{T}_{\text{polar}}(r, \theta) = (r\cos\theta, r\sin\theta).$$

is so common you may as well memorize its Jacobian:

$$J_{\mathbf{T}} = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial r}(r\sin\theta) \\ \frac{\partial}{\partial \theta}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}.$$

$$|\det J_{\mathbf{T}}| = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r.$$

So by change-of-variables (covered tomorrow), we get the following. This particular change is so common you should memorize it so you don't have to re-compute the Jacobian above all the time.

Memorize: Change-of-variables Jacobian for polar coordinates

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Many other sources will write $\mathrm{d}A$ as a shorthand for *both*: so if you have Cartesian (x, y) coordinates then $\mathrm{d}A := \mathrm{d}x\,\mathrm{d}y$, while if you have polar coordinates then $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. (They're equal, after all.) If you are worried about forgetting the factor of r, you can write $\mathrm{d}x\,\mathrm{d}y$ or $\mathrm{d}y\,\mathrm{d}x$ you know you're *supposed* to make a change of variables (and won't accidentally write $\mathrm{d}r\,\mathrm{d}\theta$ with the factor of r missing). See LAMV §25 for more on shorthand.

§3 Recipes (see LAMV §22 for details)

Recipe for total mass and center of mass

Suppose $\mathcal{R}$ is a region and ρ is a density function for the region. The total mass is given by $\operatorname{mass}(\mathcal{R}) = \iint_{\mathcal{R}} \rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$. The center of mass is given by the point

$$(\bar{x}, \bar{y}) := \left(\frac{\iint_{\mathcal{R}} x \cdot \rho(x, y)\,\mathrm{d}x\,\mathrm{d}y}{\operatorname{mass}(\mathcal{R})}, \frac{\iint_{\mathcal{R}} y \cdot \rho(x, y)\,\mathrm{d}x\,\mathrm{d}y}{\operatorname{mass}(\mathcal{R})} \right).$$

Recipe for swapping the order of integration

Given $\int_{x=?}^{?} \int_{y=?}^{?} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$, to switch the order of integration the other way:

1. Convert the limits of integration *back* into inequality format, writing the region $\mathcal{R}$.
2. Re-apply the recipe from last recitation using the other variable as the outer one now.

Also, check §24.2 LAMV (example on offset circle) for one common shape that shows up a lot in problem sets / exams (e.g. in Q3/Q4 below and at least one pset question, albeit rotated).

§4 Recitation questions from official course

1. Evaluate[1] the iterated integral $\int_{y=0}^{2} \int_{x=y/2}^{1} e^{-x^2}\,\mathrm{d}A$.

2. Evaluate the integral

$$\iint_D \frac{dA}{3 + x^2 + y^2}$$

where D is the region such that $x \geq 0$, $y \geq 0$ and $x^2 + y^2 \leq 9$.

3. A shape has a density given by $\delta(x, y) = \sqrt{x^2 + y^2}$. What is the mass of the shape defined by $x^2 + (y-1)^2 \leq 1$?

4. Evaluate the integral

$$\iint_D \frac{dA}{\sqrt{1 - x^2 - y^2}}$$

where D is the region such that $x \geq 0$ and $x^2 + (y - 1/2)^2 \leq 1/4$.

[1] Note the real course will probably say $\int_0^2 \int_{y/2}^1 e^{-x^2}\,\mathrm{d}x\,\mathrm{d}y$ and make you read backwards.