

Quick answer key to Recitation 13

ChatGPT 4o

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Use the table of contents below to skip to a specific part without seeing spoilers to the other parts.

I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

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1 Problem

Calculate the double integral of the function $f(x, y) = 6x^2 + 2y$ over the rectangle $R = [0, 2] \times [-1, 1]$. Use both vertical and horizontal slicings and check you get the same answer.

2 Solution

We are tasked with computing the double integral of $f(x, y) = 6x^2 + 2y$ over the rectangle $R = [0, 2] \times [-1, 1]$. We will solve this integral using both vertical and horizontal slicings.

2.1 1. Vertical Slicing (Integrating over y first)

For vertical slicing, we first integrate with respect to y and then with respect to x . The double integral is:

$$\int_0^2 \int_{-1}^1 (6x^2 + 2y) dy dx$$

First, compute the inner integral (with respect to y):

$$\int_{-1}^1 (6x^2 + 2y) dy = \int_{-1}^1 6x^2 dy + \int_{-1}^1 2y dy$$

For the first term:

$$\int_{-1}^1 6x^2 dy = 6x^2 \cdot (y \Big|_{-1}^1) = 6x^2 \cdot (1 - (-1)) = 6x^2 \cdot 2 = 12x^2$$

For the second term:

$$\int_{-1}^1 2y dy = 2 \cdot \left(\frac{y^2}{2} \Big|_{-1}^1 \right) = 2 \cdot \left(\frac{1^2}{2} - \frac{(-1)^2}{2} \right) = 2 \cdot \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

Thus, the inner integral simplifies to:

$$\int_{-1}^1 (6x^2 + 2y) dy = 12x^2$$

Now, we integrate with respect to x :

$$\int_0^2 12x^2 dx = 12 \cdot \int_0^2 x^2 dx = 12 \cdot \left(\frac{x^3}{3} \Big|_0^2 \right) = 12 \cdot \frac{8}{3} = 32$$

So, the value of the double integral using vertical slicing is 32.

2.2 2. Horizontal Slicing (Integrating over x first)

For horizontal slicing, we first integrate with respect to x and then with respect to y . The double integral is:

$$\int_{-1}^1 \int_0^2 (6x^2 + 2y) dx dy$$

First, compute the inner integral (with respect to x):

$$\int_0^2 (6x^2 + 2y) dx = \int_0^2 6x^2 dx + \int_0^2 2y dx$$

For the first term:

$$\int_0^2 6x^2 dx = 6 \cdot \left(\frac{x^3}{3} \Big|_0^2 \right) = 6 \cdot \frac{8}{3} = 16$$

For the second term:

$$\int_0^2 2y dx = 2y \cdot \left(x \Big|_0^2 \right) = 2y \cdot (2 - 0) = 4y$$

Thus, the inner integral simplifies to:

$$\int_0^2 (6x^2 + 2y) dx = 16 + 4y$$

Now, we integrate with respect to y :

$$\int_{-1}^1 (16 + 4y) dy = \int_{-1}^1 16 dy + \int_{-1}^1 4y dy$$

For the first term:

$$\int_{-1}^1 16 dy = 16 \cdot \left(y \Big|_{-1}^1 \right) = 16 \cdot (1 - (-1)) = 16 \cdot 2 = 32$$

For the second term:

$$\int_{-1}^1 4y dy = 4 \cdot \left(\frac{y^2}{2} \Big|_{-1}^1 \right) = 4 \cdot \left(\frac{1^2}{2} - \frac{(-1)^2}{2} \right) = 4 \cdot \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

Thus, the total value of the integral is:

$$\int_{-1}^1 (16 + 4y) dy = 32$$

So, the value of the double integral using horizontal slicing is also 32.

3 Conclusion

Both vertical and horizontal slicings give the same result. The value of the double integral is:

$$\boxed{32}$$

4 Problem

Let R be the first-quadrant region bounded by the two curves $y = \sqrt{x}$ and $y = x^3$. Compute in two different ways the double integral:

$$\iint_R xy^2 dA$$

5 Solution

I think ChatGPT is just completely wrong here LOL. Not even going to paste that output.

6 Problem

Let R be the (bounded) region between the parabola $y^2 = x$ and the line through $(2, 0)$ having slope 1. Find the points where the curves intersect and describe the region R in terms of horizontal and vertical slices. Express the double integral $\iint_R f(x, y) dA$ as an iterated integral in both ways, using both horizontal and vertical slicings. In the second case, you will have to write the integral in two pieces.

7 Solution

We are given two curves: the parabola $y^2 = x$ and the line through $(2, 0)$ with slope 1, which is given by $y = x - 2$.

7.1 1. Points of Intersection

To find the points where these curves intersect, we set $x = y^2$ (from the parabola equation) equal to $x = y + 2$ (from the line equation):

$$y^2 = y + 2$$

Rearrange this into a standard quadratic form:

$$y^2 - y - 2 = 0$$

Factor the quadratic:

$$(y - 2)(y + 1) = 0$$

Thus, the solutions are $y = 2$ and $y = -1$.

For $y = 2$, substitute into $y^2 = x$ to find $x = 4$. For $y = -1$, substitute into $y^2 = x$ to find $x = 1$.

Therefore, the curves intersect at the points $(4, 2)$ and $(1, -1)$.

7.2 2. Describing the Region

The region R is bounded on the left by the parabola $y^2 = x$ and on the right by the line $y = x - 2$, between the intersection points $(1, -1)$ and $(4, 2)$.

7.3 3. Horizontal Slicing

For horizontal slicing, we treat y as the independent variable. The limits for x at each y are from the parabola $x = y^2$ (left boundary) to the line $x = y + 2$ (right boundary). The limits for y are from -1 to 2 . Thus, the double integral is:

$$\iint_R f(x, y) dA = \int_{-1}^2 \int_{y^2}^{y+2} f(x, y) dx dy$$

7.4 4. Vertical Slicing

For vertical slicing, we treat x as the independent variable. The region is split into two parts based on the curves. For $x \in [1, 4]$, the lower boundary is $y = -\sqrt{x}$ and the upper boundary is $y = \sqrt{x}$ (from the parabola equation). For $x \in [0, 1]$, the upper boundary is instead determined by the line $y = x - 2$.

Thus, the double integral must be split into two parts:

1. For $x \in [0, 1]$, the upper boundary is $y = \sqrt{x}$ and the lower boundary is $y = -\sqrt{x}$:

$$\int_1^2 \int_{-\sqrt{x}}^{\sqrt{x}} f(x, y) dy dx$$

2. For $x \in [1, 4]$, the upper boundary is $y = x - 2$ and the lower boundary is $y = -\sqrt{x}$:

$$\int_2^4 \int_{-\sqrt{x}}^{x-2} f(x, y) dy dx$$

8 Conclusion

Thus, the double integral can be expressed in two ways:

$$\iint_R f(x, y) dA = \int_{-1}^2 \int_{y^2}^{y+2} f(x, y) dx dy$$

(using horizontal slicing)

or

$$\iint_R f(x, y) dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} f(x, y) dy dx + \int_1^4 \int_{-\sqrt{x}}^{x-2} f(x, y) dy dx$$

(using vertical slicing in two parts).