# Quick answer key to R12

ChatGPT 4o

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

## Contents

# 1 Solution

We are given the function:

$$f(x, y) = x^3 - 3xy + y^3.$$

We are tasked with finding the critical points and using the second derivative test to classify them.

## 1.1 Step 1: Find the critical points

To find the critical points, we first compute the partial derivatives of $f(x, y)$ with respect to $x$ and $y$.

**Partial derivative with respect to $x$:**

$$f_x = \frac{\partial}{\partial x}\left(x^3 - 3xy + y^3\right) = 3x^2 - 3y.$$

**Partial derivative with respect to $y$:**

$$f_y = \frac{\partial}{\partial y}\left(x^3 - 3xy + y^3\right) = -3x + 3y^2.$$

We set both partial derivatives equal to zero to find the critical points:

$$f_x = 3x^2 - 3y = 0 \quad \Rightarrow \quad x^2 = y.$$
$$f_y = -3x + 3y^2 = 0 \quad \Rightarrow \quad x = y^2.$$

## 1.2 Step 2: Solve the system of equations

We substitute $y = x^2$ (from $x^2 = y$) into the equation $x = y^2$ to find the values of $x$ and $y$.

Substitute $y = x^2$ into $x = y^2$:

$$x = (x^2)^2 = x^4.$$

Solve for $x$:

$$x^4 - x = 0 \quad \Rightarrow \quad x(x^3 - 1) = 0.$$

Thus, $x = 0$ or $x = 1$.

**Case 1:** $x = 0$ Substitute $x = 0$ into $y = x^2$:

$$y = 0^2 = 0.$$

Thus, $(0, 0)$ is a critical point.

**Case 2:** $x = 1$ Substitute $x = 1$ into $y = x^2$:

$$y = 1^2 = 1.$$

Thus, $(1, 1)$ is another critical point.

In summary, the critical points are $(0, 0)$ and $(1, 1)$.

### 1.3 Step 3: Use the second derivative test to classify the critical points

To classify the critical points, we use the second derivative test. First, compute the second partial derivatives of $f(x, y)$.

**Second partial derivatives:**

$$f_{xx} = \frac{\partial}{\partial x}\left(3x^2 - 3y\right) = 6x,$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-3x + 3y^2\right) = 6y,$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3x^2 - 3y\right) = -3.$$

The second derivative test is based on the Hessian determinant $H$, which is given by:

$$H = f_{xx}f_{yy} - (f_{xy})^2.$$

We evaluate $H$ at each critical point.

### 1.3.1 At $(0, 0)$:

At $(0, 0)$, the second partial derivatives are:

$$f_{xx}(0, 0) = 6(0) = 0, \quad f_{yy}(0, 0) = 6(0) = 0, \quad f_{xy}(0, 0) = -3.$$

The Hessian determinant is:

$$H(0, 0) = (0)(0) - (-3)^2 = -9.$$

Since $H(0, 0) < 0$, the point $(0, 0)$ is a **saddle point**.

### 1.3.2 At $(1, 1)$:

At $(1, 1)$, the second partial derivatives are:

$$f_{xx}(1, 1) = 6(1) = 6, \quad f_{yy}(1, 1) = 6(1) = 6, \quad f_{xy}(1, 1) = -3.$$

The Hessian determinant is:

$$H(1, 1) = (6)(6) - (-3)^2 = 36 - 9 = 27.$$

Since $H(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, the point $(1, 1)$ is a **local minimum**.

## 1.4 Conclusion

The critical points of the function $f(x, y) = x^3 - 3xy + y^3$ are $(0, 0)$ and $(1, 1)$. Using the second derivative test: - The point $(0, 0)$ is a **saddle point**. - The point $(1, 1)$ is a **local minimum**.

## 2 Solution

We are tasked with finding the maximum and minimum values of the function:

$$f(x, y) = x^3 + y^3$$

on the region defined by the constraint:

$$x^2 + 2y^2 \leq 36.$$

This region is an ellipse centered at the origin.

## 2.1 Step 1: Analyze the constraint and region

The region defined by $x^2 + 2y^2 \leq 36$ is an ellipse. To parameterize the boundary, we rewrite the equation of the ellipse as:

$$\frac{x^2}{36} + \frac{y^2}{18} = 1.$$

Thus, the semi-major axis is along the $x$-axis with length 6, and the semi-minor axis is along the $y$-axis with length $\sqrt{18} \approx 4.24$.

## 2.2 Step 2: Use the method of Lagrange multipliers

To find the maximum and minimum values of $f(x,y) = x^3 + y^3$ on the boundary of the region, we use the method of Lagrange multipliers. The constraint function is:

$$g(x,y) = x^2 + 2y^2 - 36 = 0.$$

The gradients of $f$ and $g$ are:

$$\nabla f(x,y) = \left\langle 3x^2, 3y^2 \right\rangle,$$

$$\nabla g(x,y) = \langle 2x, 4y \rangle.$$

According to the method of Lagrange multipliers, we must have:

$$\nabla f(x,y) = \lambda \nabla g(x,y).$$

This gives the system of equations:

$$3x^2 = \lambda(2x),$$

$$3y^2 = \lambda(4y).$$

**Case 1:** $x = 0$ Substitute $x = 0$ into the constraint $x^2 + 2y^2 = 36$:

$$0^2 + 2y^2 = 36 \quad \Rightarrow \quad y^2 = 18 \quad \Rightarrow \quad y = \pm\sqrt{18} = \pm 3\sqrt{2}.$$

At $(0, 3\sqrt{2})$ and $(0, -3\sqrt{2})$, the function $f(x,y)$ becomes:

$$f(0, 3\sqrt{2}) = 0^3 + (3\sqrt{2})^3 = 0 + 54\sqrt{2},$$

$$f(0, -3\sqrt{2}) = 0^3 + (-3\sqrt{2})^3 = 0 - 54\sqrt{2}.$$

**Case 2:** $y = 0$ Substitute $y = 0$ into the constraint $x^2 + 2y^2 = 36$:

$$x^2 + 0^2 = 36 \quad \Rightarrow \quad x^2 = 36 \quad \Rightarrow \quad x = \pm 6.$$

At $(6, 0)$ and $(-6, 0)$, the function $f(x,y)$ becomes:

$$f(6,0) = 6^3 + 0^3 = 216,$$

$$f(-6,0) = (-6)^3 + 0^3 = -216.$$

**Case 3: $x \neq 0$ and $y \neq 0$** For $x \neq 0$ and $y \neq 0$, we can solve for $\lambda$ from the Lagrange multiplier equations:

$$3x^2 = \lambda(2x) \quad \Rightarrow \quad \lambda = \frac{3x}{2},$$

$$3y^2 = \lambda(4y) \quad \Rightarrow \quad \lambda = \frac{3y^2}{4y} = \frac{3y}{4}.$$

Equating the two expressions for $\lambda$:

$$\frac{3x}{2} = \frac{3y}{4} \quad \Rightarrow \quad 4x = 2y \quad \Rightarrow \quad y = 2x.$$

Substitute $y = 2x$ into the constraint $x^2 + 2y^2 = 36$:

$$x^2+2(2x)^2 = 36 \quad \Rightarrow \quad x^2+8x^2 = 36 \quad \Rightarrow \quad 9x^2 = 36 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2.$$

When $x = 2$, $y = 4$, and when $x = -2$, $y = -4$.

At $(2, 4)$ and $(-2, -4)$, the function $f(x, y)$ becomes:

$$f(2, 4) = 2^3 + 4^3 = 8 + 64 = 72,$$

$$f(-2, -4) = (-2)^3 + (-4)^3 = -8 - 64 = -72.$$

### 2.3 Step 4: Evaluate the values

We summarize the values of $f(x, y)$ at the critical points:

- $f(6, 0) = 216$ - $f(-6, 0) = -216$ - $f(0, 3\sqrt{2}) = 54\sqrt{2} \approx 76.37$ - $f(0, -3\sqrt{2}) = -54\sqrt{2} \approx -76.37$ - $f(2, 4) = 72$ - $f(-2, -4) = -72$

### 2.4 Step 5: Conclusion

The **global maximum** value of $f(x, y)$ on the region $x^2 + 2y^2 \leq 36$ is:

$$f_{\text{max}} = 216 \text{ at } (6, 0).$$

The **global minimum** value of $f(x, y)$ on the region $x^2 + 2y^2 \leq 36$ is:

$$f_{\text{min}} = -216 \text{ at } (-6, 0).$$

## 3 Solution

ChatGPT go this wrong. Kappa.

## 4 Solution

We are tasked with finding the maximum and minimum values of the function:

$$f(x, y, z) = x^2 + x + 3y^2 + 2z^2$$

subject to the constraint:

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

## 4.1 Step 1: Set up the Lagrangian

We use the method of Lagrange multipliers by defining the Lagrangian function:

$$\mathcal{L}(x, y, z, \lambda) = x^2 + x + 3y^2 + 2z^2 - \lambda(x^2 + y^2 + z^2 - 1).$$

### 4.2 Step 2: Compute the partial derivatives

Compute the partial derivatives of $\mathcal{L}$ with respect to $x$, $y$, $z$, and $\lambda$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 2x + 1 - 2\lambda x = 0, \\
\frac{\partial \mathcal{L}}{\partial y} &= 6y - 2\lambda y = 0, \\
\frac{\partial \mathcal{L}}{\partial z} &= 4z - 2\lambda z = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -(x^2 + y^2 + z^2 - 1) = 0.
\end{aligned}$$

### 4.3 Step 3: Solve the system of equations

We have the following system:

$$2x + 1 - 2\lambda x = 0 \quad (1) \tag{1}$$

$$6y - 2\lambda y = 0 \quad (2) \tag{2}$$

$$4z - 2\lambda z = 0 \quad (3) \tag{3}$$

$$x^2 + y^2 + z^2 = 1 \quad (4) \tag{4}$$

**From Equation (2):** Either $y = 0$ or $6 - 2\lambda = 0$.
If $y \neq 0$, then:

$$6 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = 3.$$

**From Equation (3):** Either $z = 0$ or $4 - 2\lambda = 0$.
If $z \neq 0$, then:

$$4 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = 2.$$

Since $\lambda$ cannot be both 3 and 2, we consider the cases where either $y = 0$ or $z = 0$.

### 4.3.1 Case 1: $y = 0$

From Equation (3):

$$4z - 2\lambda z = 0 \quad \Rightarrow \quad z(4 - 2\lambda) = 0.$$

**Subcase 1.1:** $z = 0$
Then from Equation (1):

$$2x + 1 - 2\lambda x = 0 \quad \Rightarrow \quad 2x(1 - \lambda) + 1 = 0.$$

**Subcase 1.1.1: Solve for** $x$
If $x \neq 0$, then:

$$2x(1 - \lambda) + 1 = 0 \quad \Rightarrow \quad x = \frac{-1}{2(1 - \lambda)}.$$

But since $x^2 + y^2 + z^2 = 1$ and $y = 0$, $z = 0$, we have:

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1.$$

**Subcase 1.1.2: Find** $\lambda$ **for** $x = \pm 1$
For $x = 1$:

$$2(1) + 1 - 2\lambda(1) = 0 \quad \Rightarrow \quad 2 + 1 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{3}{2}.$$

For $x = -1$:

$$2(-1) + 1 - 2\lambda(-1) = 0 \quad \Rightarrow \quad -2 + 1 + 2\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2}.$$

**Subcase 1.2:** $z \neq 0$
Then $4 - 2\lambda = 0 \quad \Rightarrow \quad \lambda = 2$.
From Equation (1):

$$2x + 1 - 4x = 0 \quad \Rightarrow \quad -2x + 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}.$$

From Equation (4):

$$\left(\frac{1}{2}\right)^2 + 0 + z^2 = 1 \quad \Rightarrow \quad z^2 = \frac{3}{4} \quad \Rightarrow \quad z = \pm\frac{\sqrt{3}}{2}.$$

#### 4.3.2 Case 2: $z = 0$

Similarly, we find $y = \pm\frac{\sqrt{15}}{4}$ and $x = \frac{1}{4}$.

### 4.4 Step 4: Evaluate $f(x, y, z)$ at critical points

**Point 1:** $(x, y, z) = (1, 0, 0)$

$$f(1, 0, 0) = (1)^2 + 1 + 0 + 0 = 2.$$

**Point 2:** $(x, y, z) = (-1, 0, 0)$

$$f(-1, 0, 0) = (-1)^2 + (-1) + 0 + 0 = 1 - 1 = 0.$$

**Point 3:** $\left(\frac{1}{2}, 0, \pm\frac{\sqrt{3}}{2}\right)$

$$f\left(\frac{1}{2}, 0, \pm\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}\right)^2 + \frac{1}{2} + 0 + 2\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{1}{2} + 0 + \frac{3}{2} = \frac{9}{4} = 2.25.$$

**Point 4:** $\left(\frac{1}{4}, \pm\frac{\sqrt{15}}{4}, 0\right)$

$$f\left(\frac{1}{4}, \pm\frac{\sqrt{15}}{4}, 0\right) = \left(\frac{1}{4}\right)^2 + \frac{1}{4} + 3\left(\frac{\sqrt{15}}{4}\right)^2 + 0 = \frac{1}{16} + \frac{1}{4} + 3\left(\frac{15}{16}\right) = \frac{1}{16} + \frac{1}{4} + \frac{45}{16} = \frac{50}{16} = \frac{25}{8} = 3.125.$$

### 4.5 Step 5: Determine the maximum and minimum values

From the evaluations, we have:

- $f_{\text{max}} = \dfrac{25}{8} = 3.125$ at $\left(\dfrac{1}{4}, \pm\dfrac{\sqrt{15}}{4}, 0\right)$.
- $f_{\text{min}} = 0$ at $(-1, 0, 0)$.

### 4.6 Conclusion

The maximum value of $f(x, y, z)$ on the unit sphere is $\frac{25}{8}$ at the points $\left(\frac{1}{4}, \pm\frac{\sqrt{15}}{4}, 0\right)$. The minimum value is 0 at the point $(-1, 0, 0)$.