Notes for 18.02 Recitation 12

18.02 Recitation MW9

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At some points, I was not convinced this puzzle should even exist, but editors and testsolvers encouraged me to finish it, saying it was "exactly the bullshit I expect to see in Mystery Hunt". (The lucky postprodder for the puzzle would also like to voice his support.) Hope you had fun! Really, it could have been much worse.

— Alex Irpan in the author notes for the Mystery Hunt puzzle 5D Barred Diagramless with Multiverse Time Travel

This handout (and any other DLC's I write) are posted at `https://web.evanchen.cc/1802.html`.

§1 Announcements and the like

Welcome back from the long weekend!

- **Midterm 2** is probably up to Lagrange multipliers but not including tomorrow's class.
- **Midterm 2 review session** is scheduled 3pm-5pm in 4-370.
  - It will be a mock exam 3pm-4pm then solution presentation by me for 4pm-5pm.
  - I will post the exam beforehand, so if you can only make 4pm-5pm you can try the mock yourself and then come to solutions. Solutions posted later too.
- **LAMV** is updated in preparation for Midterm 2, corresponds to Parts Delta, Echo, Foxtrot. (This is the mega-file titled "Linear Algebra and Multivariable Calculus" on my page.)
  - Part Delta has all the parametric stuff, it's about halfway written.
  - Part Echo has all the partial derivative and gradient material and is essentially done.
  - Part Foxtrot has all the critical point and Lagrange multipliers stuff and is essentially done.

§2 Highlights from LAMV

Again, see the full book at `https://web.evanchen.cc/upload/1802/lamv.pdf` for full exposition — (much more than I can fit on one printed page! But here is a quick cheat sheet.

For every region, you care about what LAMV calls **boundary**, **limit case**, and **dimension**.

Recipe: The rule of thumb for regions

- The dimension is probably n minus the number of $=$ constraints.
- The limit cases are obtained by turning $<$ and $>$ into limits, and considering when any of the variables can go to $\pm\infty$.
- The boundary is obtained when any $\leq$ and $\geq$ becomes $=$.

Constraint	Boundary	Limit case	Dimension
$\leq$ or $\geq$	Change to $=$ to get boundary	No effect	No effect
$<$ or $>$	No effect	Approach for limit case	No effect
$=$	No effect	No effect	Reduces dim by one

Table 1: Rules of thumb.

For 18.02, we say region $\mathcal{R}$ is compact if there are no limit cases. That is, all the constraints are $=$, $\leq$, or $\geq$ (no $<$ or $>$) and none of the variables can go to $\pm\infty$.

Tip: Compact optimization theorem

If $\mathcal{R}$ is a compact region, and f is a function to optimize on the region which is continuously defined everywhere, then there must be at least one global minimum, and at least one global maximum. (Works in both no-LM and LM case.)

Recipe for optimization without Lagrange Multipliers

Suppose you want to find the optimal values of $f : \mathbb{R}^n \to \mathbb{R}$ over a region $\mathcal{R}$, and $\mathcal{R}$ has dimension n.

1. Evaluate f on all the **critical points** of f in the region $\mathcal{R}$.
2. Evaluate f on all the **boundary points** of f in the region $\mathcal{R}$.
3. Evaluate f on all the **limit cases** of f in the region $\mathcal{R}$.
4. Output the points in the previous steps that give the best values, or assert the optimal value doesn't exist (if points in step 3 do better than steps 1-2).

Recipe for Lagrange multipliers

Suppose you want to find the optimal values of $f : \mathbb{R}^n \to \mathbb{R}$ over a region $\mathcal{R}$, and $\mathcal{R}$ has dimension $n-1$ due to a single constraint $g = c$ for some $g : \mathbb{R}^n \to \mathbb{R}$.

1. Evaluate f on all the **LM-critical points** of f that lie on the region $\mathcal{R}$.
2. Evaluate f on all the **boundary points** of f of the region $\mathcal{R}$.
3. Evaluate f on all the **limit cases** of f of the region $\mathcal{R}$.
4. Output the points in the previous steps that give the best values, or assert the optimal value doesn't exist (if points in step 3 do better than steps 1-2).

If there are any points at which ∇f, ∇g are undefined, you should check those as well. (Rare.)

§3 Recitation questions from official course

(Exercise before you start: which of these questions does compact optimization theorem apply?)

1. Consider the function $f(x, y) = x^3 - 3xy + y^3$. Find the critical points of the function and use the second derivative test to classify them.

2. Find the maximum and minimum values of $x^3 + y^3$ on the region given by $x^2 + 2y^2 \leq 36$.

3. Find the maximum and minimum of $x^2 + x + 3y^2 + 2z^2$ on the unit sphere $x^2 + y^2 + z^2 = 1$.