Quick answer key to Recitation 10

ChatGPT 4o

7 October 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution

We are given the function $f(x, y) = \frac{1}{x^2+y^2}$. We will go through each part of the question step by step.

1.1 Part 1: Draw the level curves for $f(x, y) = 1$ and $f(x, y) = 2$

The level curves for the function are obtained by setting $f(x, y) = c$ for some constant c, i.e.:

$$f(x, y) = \frac{1}{x^2 + y^2} = c$$

$$x^2 + y^2 = \frac{1}{c}$$

For $f(x, y) = 1$, we have:

$$x^2 + y^2 = 1$$

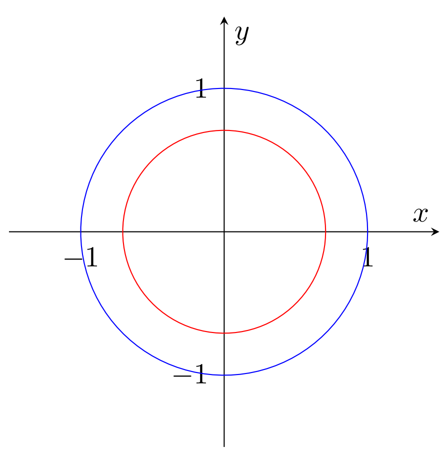
This is the equation of a circle with radius 1.

For $f(x, y) = 2$, we have:

$$x^2 + y^2 = \frac{1}{2}$$

This is the equation of a circle with radius $\frac{1}{\sqrt{2}}$.

The level curves are circles centered at the origin with different radii. Below is the plot of the level curves:



The blue circle represents the level curve for $f(x, y) = 1$, and the red circle represents the level curve for $f(x, y) = 2$.

1.2 Part 2: Sketch the graph of $f(x, y)$

The function $f(x, y) = \frac{1}{x^2+y^2}$ is a surface that decreases as $x^2 + y^2$ increases. The surface is radially symmetric and decreases toward 0 as we move away from the origin.

See WolframAlpha for a picture.

1.3 Part 3: Find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$

The partial derivative of $f(x, y)$ with respect to x is:

$$f_x(x, y) = \frac{\partial}{\partial x}\left(\frac{1}{x^2 + y^2}\right)$$

Using the chain rule:

$$f_x(x, y) = -\frac{1}{(x^2 + y^2)^2} \cdot 2x = -\frac{2x}{(x^2 + y^2)^2}$$

Similarly, the partial derivative with respect to y is:

$$f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{1}{x^2 + y^2}\right) = -\frac{2y}{(x^2 + y^2)^2}$$

Thus, the partial derivatives are:

$$f_x(x, y) = -\frac{2x}{(x^2 + y^2)^2}, \quad f_y(x, y) = -\frac{2y}{(x^2 + y^2)^2}$$

1.4 Part 4: In what direction does $f(x, y)$ increase the fastest at $(1, 2)$? What is the rate of increase in this direction?

The direction in which $f(x, y)$ increases the fastest is given by the gradient of $f(x, y)$:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

At the point $(1, 2)$, the gradient is:

$$\nabla f(1, 2) = \left\langle -\frac{2(1)}{(1^2 + 2^2)^2}, -\frac{2(2)}{(1^2 + 2^2)^2} \right\rangle = \left\langle -\frac{2}{25}, -\frac{4}{25} \right\rangle$$

To get the unit vector direction $\mathbf{u}$ in which $f(x, y)$ increases the fastest, we normalize the gradient vector:

$$\mathbf{u} = \frac{\nabla f(1, 2)}{|\nabla f(1, 2)|} = \frac{1}{\sqrt{\left(-\frac{2}{25}\right)^2 + \left(-\frac{4}{25}\right)^2}} \left\langle -\frac{2}{25}, -\frac{4}{25} \right\rangle$$

$$= \frac{25}{5} \left\langle -\frac{2}{25}, -\frac{4}{25} \right\rangle = \left\langle -\frac{2}{5}, -\frac{4}{5} \right\rangle$$

The rate of increase in this direction is the magnitude of the gradient:

$$|\nabla f(1, 2)| = \sqrt{\left(-\frac{2}{25}\right)^2 + \left(-\frac{4}{25}\right)^2} = \frac{1}{5}$$

Thus, the rate of increase in this direction is $\frac{1}{5}$.

1.5 Part 5: Find the directional derivative $D_{\mathbf{u}}f(1,2)$ for the unit vector $\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$

The directional derivative in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is given by:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = f_x(x,y)u_1 + f_y(x,y)u_2$$

At $(1,2)$, we have:

$$D_{\mathbf{u}}f(1,2) = \left(-\frac{2}{25}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(-\frac{4}{25}\right)\left(-\frac{2}{\sqrt{5}}\right)$$

$$= -\frac{2}{25\sqrt{5}} + \frac{8}{25\sqrt{5}} = \frac{6}{25\sqrt{5}}$$

Thus, the directional derivative is:

$$D_{\mathbf{u}}f(1,2) = \frac{6}{25\sqrt{5}}$$

2 Solution

We are tasked with estimating the value of $\log(0.49^2 + 0.76)$ using the linear approximation of the function $f(x, y) = \log(x^2 + y)$ near the point $(x_0, y_0) = (0.5, 0.75)$.

2.1 Step 1: Compute the function value at $(0.5, 0.75)$

The function is $f(x, y) = \log(x^2 + y)$. At the point $(x_0, y_0) = (0.5, 0.75)$:

$$f(0.5, 0.75) = \log(0.5^2 + 0.75) = \log(0.25 + 0.75) = \log(1) = 0.$$

2.2 Step 2: Compute the partial derivatives of $f(x, y)$

The partial derivatives of $f(x, y) = \log(x^2 + y)$ are:

$$f_x(x, y) = \frac{\partial}{\partial x}\left(\log(x^2 + y)\right) = \frac{1}{x^2 + y} \cdot \frac{\partial}{\partial x}(x^2 + y) = \frac{2x}{x^2 + y},$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(\log(x^2 + y)\right) = \frac{1}{x^2 + y} \cdot \frac{\partial}{\partial y}(x^2 + y) = \frac{1}{x^2 + y}.$$

2.3 Step 3: Evaluate the partial derivatives at $(0.5, 0.75)$

Now, we evaluate the partial derivatives at the point $(0.5, 0.75)$:

$$f_x(0.5, 0.75) = \frac{2(0.5)}{0.5^2 + 0.75} = \frac{1}{1} = 1,$$

$$f_y(0.5, 0.75) = \frac{1}{0.5^2 + 0.75} = \frac{1}{1} = 1.$$

2.4 Step 4: Use the linear approximation formula

The linear approximation of the function $f(x, y)$ near the point $(x_0, y_0) = (0.5, 0.75)$ is given by:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Using the values we have calculated:

$$f(x, y) \approx 0 + 1(x - 0.5) + 1(y - 0.75).$$

Thus, the linear approximation is:

$$f(x, y) \approx (x - 0.5) + (y - 0.75).$$

2.5 Step 5: Apply the approximation to $(x, y) = (0.49, 0.76)$

We now use the linear approximation to estimate $f(0.49, 0.76)$:

$$f(0.49, 0.76) \approx (0.49 - 0.5) + (0.76 - 0.75) = -0.01 + 0.01 = 0.$$

2.6 Final Estimate

Using the linear approximation, we estimate that:

$$\log(0.49^2 + 0.76) \approx 0.$$

Thus, the value of $\log(0.49^2 + 0.76)$ is approximately 0 based on the linear approximation.

3 Solution

We are given the surface S defined by the equation:

$$xy - 2xz^2 + 3y^2z = 2$$

This defines a level surface for the function $f(x, y, z) = xy - 2xz^2 + 3y^2z$, with the equation $f(x, y, z) = 2$. We are asked to find the directional derivative of $f(x, y, z)$ at the point $(1, 1, 1)$ in the direction of a vector $\mathbf{u}$ that is tangent to the surface S at this point. Additionally, we are asked to find the equation of the tangent plane to S at the point $(1, 1, 1)$.

3.1 Part 1: Directional Derivative in the Direction of a Tangent Vector

To find the directional derivative of $f(x, y, z)$ at the point $(1, 1, 1)$ in the direction of a tangent vector $\mathbf{u}$, we first compute the gradient of $f(x, y, z)$. The gradient of f is:

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

We compute each partial derivative:

$$\frac{\partial f}{\partial x} = y - 2z^2,$$

$$\frac{\partial f}{\partial y} = x + 6yz,$$

$$\frac{\partial f}{\partial z} = -4xz + 3y^2.$$

Now, evaluate the gradient at the point $(1, 1, 1)$:

$$\nabla f(1, 1, 1) = \left\langle 1 - 2(1)^2, 1 + 6(1)(1), -4(1)(1) + 3(1)^2 \right\rangle = \langle -1, 7, -1 \rangle.$$

Let $\mathbf{u}$ be any vector tangent to the surface S at the point $(1, 1, 1)$. Since $\mathbf{u}$ is tangent to the surface, it must be perpendicular to the gradient $\nabla f(1, 1, 1)$. Therefore, the dot product between $\mathbf{u}$ and $\nabla f(1, 1, 1)$ is:

$$\nabla f(1, 1, 1) \cdot \mathbf{u} = 0.$$

Since the directional derivative of $f(x, y, z)$ in the direction of a vector tangent to the surface is the dot product of the gradient and the tangent vector, the directional derivative in the direction of **u** is:

$$D_{\mathbf{u}} f(1,1,1) = \nabla f(1,1,1) \cdot \mathbf{u} = 0.$$

Thus, the directional derivative of $f(x, y, z)$ at $(1, 1, 1)$ in the direction of a tangent vector **u** is zero.

3.2 Part 2: Equation of the Tangent Plane to the Surface at $(1, 1, 1)$

The equation of the tangent plane to the surface at the point (x_0, y_0, z_0) is given by:

$$\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

At the point $(1, 1, 1)$, we already computed the gradient $\nabla f(1, 1, 1) = \langle -1, 7, -1 \rangle$. The equation of the tangent plane is therefore:

$$\nabla f(1, 1, 1) \cdot \langle x - 1, y - 1, z - 1 \rangle = 0.$$

This becomes:

$$-1(x - 1) + 7(y - 1) - 1(z - 1) = 0.$$

Simplifying:

$$-(x - 1) + 7(y - 1) - (z - 1) = 0,$$

$$-x + 1 + 7y - 7 - z + 1 = 0,$$

$$-x + 7y - z - 5 = 0.$$

Thus, the equation of the tangent plane to the surface S at the point $(1, 1, 1)$ is:

$$-x + 7y - z = 5.$$