

Notes for 18.02 Recitation 9

18.02 Recitation MW9

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You brought your army to my kingdom, James. Use it.

— Ozpin, in RWBY

This handout (and any other DLC's I write) are posted at <https://web.evanchen.cc/1802.html>.

§1 Keep doing everything component-wise

As I promised pretty much nothing happened yesterday and you still do everything componentwise. (Don't worry about Kepler's law.) There's one additional theorem you need.

Theorem 1.1 (Arc length). $L = \int_{t=\text{start time}}^{\text{stop time}} \|\mathbf{v}(t)\| dt$.

§2 Preview of Thursday/Friday's lecture

§2.1 Level curves replace xy -graphs

In high school algebra, you drew 2D graphs of one-variable functions like $y = 2x + 5$ or $y = x^2 + 7$. So it might have seemed a bit weird to you that in R01/R02, our planes were usually written like $2x + 5y + 3z = 7$ rather than, say, $z = \frac{7-2x-5y}{3}$. But this form turned out to be better, because it let us easily access the normal vector (which here is $\langle 2, 5, 3 \rangle$).

We'll keep up this trend often with multivariate functions; prefer constant RHS's. That is, suppose we want to draw a *two* variable function like $f(x, y) = x^2 + 3y$. Either do a 2D level curve $x^2 + 3y = c$, or think of it as the 3D level curve $x^2 + 3y - z = 0$.

§2.2 Gradients

There are two important pieces of philosophy. First: the goal of the first-order derivative is to approximate a function by a linear one. Second (from R01): *everything you used slopes for before, you should use normal vectors instead*. Here's how this plays out:

- In 18.01, when $f : \mathbb{R} \rightarrow \mathbb{R}$, you defined a **derivative** $f'(p)$ at each input $p \in \mathbb{R}$, which you thought of as the **slope** of the **tangent line** at p . Think $f(5.01) \approx f(5) + f'(5) \cdot 0.01$.
- In 18.02, when $f : \mathbb{R}^n \rightarrow \mathbb{R}$, you will instead define the **gradient** $\nabla f(P)$ for each $P \in \mathbb{R}^n$. (For *level-curve pictures*, this is a normal vector to the tangent line/plane/hyperplane.) So ∇f carries much *more* information: there are n directions to move in.

(To spell out type signatures: ∇f is a function accepting points in \mathbb{R}^n and outputting vectors in \mathbb{R}^n ; $\nabla f(P)$ is a vector in \mathbb{R}^n .)

The **dot product** is now going to be really important.¹ Here’s how.

- In 18.01, if you know $f(5)$ and $f'(5)$, then $f(5.01) \approx f(5) + f'(5) \cdot 0.01$, because we think of $f'(5)$ as a slope (“rise/run”) and 0.01 as the run.
- In 18.02, if you know $f(5, 8)$ and $\nabla f(5, 8)$, then the analogy goes

$$f(5.01, 7.996) \approx f(5, 8) + \nabla f(5, 8) \cdot \langle 0.01, -0.004 \rangle.$$

Yes, the dot is a dot product; both $\nabla f(5, 8)$ and $\langle 0.01, -0.004 \rangle$ have type “vector in \mathbb{R}^2 ”. This will take a lot of time to get used to, but everything afterwards depends on it, so do whatever you can to get this into your instincts. I’ll draw you a picture next recitation.

Thing	18.01	18.02
Function	$f : \mathbb{R} \rightarrow \mathbb{R}$	$f : \mathbb{R}^n \rightarrow \mathbb{R}$
Differentiate	$f' : \mathbb{R} \rightarrow \mathbb{R}$	$\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$
Think of as	Slope (rise/run)	Gradient (measures change in n directions)
Approximation	Multiply by small run	Dot product with small displacement

(In math-major classes like 18.100, the **total derivative** replaces the gradient. I don’t know remember if total derivative appears in 18.02 as well, but if it does, I’ll talk more about it.)

§2.3 Advice for this week

- Honestly, just ignore all this parametric stuff until MT2-ish, because basically nothing substantial happens (you just do everything component-wise) and also it won’t be reused until after MT2.
- **Start early on PSet 4B**, it’s long and introduces a lot of new concepts compared to PSet 1-3.
- **Skim through section 8 of Poonen** before Thursday and Friday lecture and thank me after. If you’re seeing this stuff for the first time, it’ll be almost impossible to keep up without pre-reading.

§3 Recitation problems from official course

- 1 (Example 10 from 12.5 of EP) A ball is thrown in the air from the origin in xyz -space. The initial velocity of the ball is $\mathbf{v}(0) = 80\mathbf{j} + 80\mathbf{k}$, measured in feet per second. The spin of the ball causes acceleration in the x -direction in addition to gravitational acceleration downward in the z -direction. The acceleration is constant:

$$\mathbf{a}(t) = 2\mathbf{i} - 32\mathbf{k}.$$

Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$. What is the speed of the ball when it hits the ground?

- 2 Suppose a particle moves along trajectory $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ where $0 \leq t \leq 2$. Calculate the total distance travelled by the particle. Do the same for the trajectory $\mathbf{r}(t) = \langle 2 \cos(3t), 2 \sin(3t) \rangle$, $0 \leq t \leq 2\pi$. In the second case, compare this with the length of the curve on the xy -plane.
- 3a Show that a particle moves at constant speed if and only if its velocity vector and acceleration vector are always perpendicular.
- 3b (from lecture) Suppose that the position vector and acceleration vector are always proportional to each other; show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is a constant vector.

¹You might have thought it was weird that back in R01 and R02, every time used the dot product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$, there was almost always this “unwanted” absolute value that we would immediately strip off. But in the gradient, we’ll actually use both the absolute values.