Notes for 18.02 Recitation 9

18.02 Recitation MW9

Evan Chen

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You brought your army to my kingdom, James. Use it.

— *Ozpin, in RWBY*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

§1 Keep doing everything component-wise

As I promised pretty much nothing happened yesterday and you still do everything componentwise. (Don't worry about Kepler's law.) There's one additional theorem you need.

Theorem 1.1 (Arc length): $L = \int_{\text{start time}}^{\text{stop time}} \|\mathbf{v}(t)\| \, dt$.

§2 Preview of Thursday/Friday's lecture

§2.1 Level curves replace xy-graphs

In high school algebra, you drew 2D graphs of one-variable functions like $y = 2x + 5$ or $y = x^2 + 7$. So it might have seemed a bit weird to you that in R01/R02, our planes were usually written like $2x + 5y + 3z = 7$ rather than, say, $z = \frac{7-2x-5y}{3}$. But this form turned out to be better, because it let us easily access the normal vector (which here is $\langle 2, 5, 3 \rangle$).

We'll keep up this trend often with multivariate functions; prefer constant RHS's. That is, suppose we want to draw a *two* variable function like $f(x, y) = x^2 + 3y$. Either do a 2D level curve $x^2 + 3y = c$, or think of it as the 3D level curve $x^2 + 3y - z = 0$.

§2.2 Gradients

There are two important pieces of philosophy. First: the goal of the first-order derivative is to approximate a function by a linear one. Second (from R01): *everything you used slopes for before, you should use normal vectors instead.* Here's how this plays out:

- In 18.01, when $f : \mathbb{R} \to \mathbb{R}$, you defined a **derivative** $f'(p)$ at each input $p \in \mathbb{R}$, which you thought of as the **slope** of the **tangent line** at p. Think $f(5.01) \approx f(5) + f'(5) \cdot 0.01$.
- In 18.02, when $f : \mathbb{R}^n \to \mathbb{R}$, you will instead define the **gradient** $\nabla f(P)$ for each $P \in \mathbb{R}^n$. (For *level-curve pictures*, this is a normal vector to the tangent line/plane/hyperplane.) So ∇f carries much *more* information: there are n directions to move in.

(To spell out type signatures: ∇f is a function accepting points in $\mathbb{R}^n$ and outputting vectors in $\mathbb{R}^n$; $\nabla f(P)$ is a vector in $\mathbb{R}^n$.)

The **dot product** is now going to be really important.[1] Here's how.

- In 18.01, if you know $f(5)$ and $f'(5)$, then $f(5.01) \approx f(5) + f'(5) \cdot 0.01$, because we think of $f'(5)$ as a slope ("rise/run") and 0.01 as the run.
- In 18.02, if you know $f(5,8)$ and $\nabla f(5,8)$, then the analogy goes

  $$f(5.01, 7.996) \approx f(5,8) + \nabla f(5,8) \cdot \langle 0.01, -0.004 \rangle.$$

  Yes, the dot is a dot product; both $\nabla f(5,8)$ and $\langle 0.01, -0.004 \rangle$ have type "vector in $\mathbb{R}^2$". This will take a lot of time to get used to, but everything afterwards depends on it, so do whatever you can to get this into your instincts. I'll draw you a picture next recitation.

Thing	18.01	18.02
Function	$f : \mathbb{R} \to \mathbb{R}$	$f : \mathbb{R}^n \to \mathbb{R}$
Differentiate	$f' : \mathbb{R} \to \mathbb{R}$	$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$
Think of as	Slope (rise/run)	Gradient (measures change in n directions)
Approximation	Multiply by small run	Dot product with small displacement

(In math-major classes like 18.100, the **total derivative** replaces the gradient. I don't know remember if total derivative appears in 18.02 as well, but if it does, I'll talk more about it.)

§2.3 Advice for this week

- Honestly, just ignore all this parametric stuff until MT2-ish, because basically nothing substantial happens (you just do everything component-wise) and also it won't be reused until after MT2.
- **Start early on PSet 4B**, it's long and introduces a lot of new concepts compared to PSet 1-3.
- **Skim through section 8 of Poonen** before Thursday and Friday lecture and thank me after. If you're seeing this stuff for the first time, it'll be almost impossible to keep up without pre-reading.

§3 Recitation problems from official course

1 (Example 10 from 12.5 of EP) A ball is thrown in the air from the origin in xyz-space. The initial velocity of the ball is $\mathbf{v}(0) = 80\mathbf{j} + 80\mathbf{k}$, measured in feet per second. The spin of the ball causes acceleration in the x-direction in addition to gravitational acceleration downward in the z-direction. The acceleration is constant:

$$\mathbf{a}(t) = 2\mathbf{i} - 32\mathbf{k}.$$

Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$. What is the speed of the ball when it hits the ground?

2 Suppose a particle moves along trajectory $\mathbf{r}(t) = \langle t^2, t^3 \rangle$ where $0 \le t \le 2$. Calculate the total distance travelled by the particle. Do the same for the trajectory $\mathbf{r}(t) = \langle 2\cos(3t), 2\sin(3t) \rangle$, $0 \le t \le 2\pi$. In the second case, compare this with the length of the curve on the xy-plane.

3a Show that a particle moves at constant speed if and only if its velocity vector and acceleration vector are always perpendicular.

3b (from lecture) Suppose that the position vector and acceleration vector are always proportional to each other; show that $\mathbf{r}(t) \times \mathbf{v}(t)$ is a constant vector.

[1] You might have thought it was weird that back in R01 and R02, every time used the dot product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta$, there was almost always this "unwanted" absolute value that we would immediately strip off. But in the gradient, we'll actually use both the absolute values.