Quick answer key to Recitation 8

ChatGPT 4o

29 September 2024

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution

We are given the following information:
- The hunter walks toward the origin along the positive x-axis with unit speed. At time $t = 0$, they are at $x = 10$.
- The rabbit moves with constant velocity $\sqrt{5}$ along the line $y = 2x$ in the first quadrant, starting at the origin at time $t = 0$.
- The hunter always aims their arrow (of unit length) toward the rabbit.

1.1 Part 1: Vector-valued function $A(t)$ for the arrow at time t

Let $A(t)$ be the vector-valued function representing the arrow's direction at time t.

Hunter's position: The hunter moves along the x-axis toward the origin with unit speed. Thus, the position of the hunter at time t is:

$$\text{Hunter's position} = \langle 10 - t, 0 \rangle$$

Rabbit's position: The rabbit moves along the line $y = 2x$ with constant velocity $\sqrt{5}$. Let the position of the rabbit at time t be $\langle x_r(t), y_r(t) \rangle$.

Since the rabbit moves along the line $y = 2x$, we have $y_r(t) = 2x_r(t)$. The rabbit's velocity $\sqrt{5}$ corresponds to the magnitude of the velocity vector:

$$\text{Velocity of the rabbit} = \left\| \frac{d}{dt} \begin{pmatrix} x_r(t) \\ y_r(t) \end{pmatrix} \right\| = \sqrt{5}$$

We know that:

$$\frac{dx_r(t)}{dt} = v_x, \quad \frac{dy_r(t)}{dt} = 2v_x$$

Using the Pythagorean theorem:

$$\sqrt{v_x^2 + (2v_x)^2} = \sqrt{5}$$

$$\sqrt{5v_x^2} = \sqrt{5} \quad \Rightarrow \quad v_x = 1$$

Thus, the rabbit's velocity components are $v_x = 1$ and $v_y = 2$. Therefore, the rabbit's position at time t is:

$$\text{Rabbit's position} = \langle t, 2t \rangle$$

Arrow direction: At any time t, the hunter aims the arrow directly at the rabbit. The vector pointing from the hunter's position to the rabbit's position is:

$$\mathbf{d}(t) = \langle t - (10 - t), 2t - 0\rangle = \langle 2t - 10, 2t\rangle$$

Now, we normalize this vector to ensure that the magnitude of the arrow's direction vector $A(t)$ is always 1. The magnitude of $\mathbf{d}(t)$ is:

$$|\mathbf{d}(t)| = \sqrt{(2t - 10)^2 + (2t)^2} = \sqrt{(4t^2 - 40t + 100) + 4t^2} = \sqrt{8t^2 - 40t + 100}$$

Thus, the normalized arrow direction vector is:

$$A(t) = \frac{\mathbf{d}(t)}{|\mathbf{d}(t)|} = \frac{\langle 2t - 10, 2t\rangle}{\sqrt{8t^2 - 40t + 100}}$$

Therefore, the vector-valued function for the arrow at time t is:

$$A(t) = \frac{\langle 2t - 10, 2t\rangle}{\sqrt{8t^2 - 40t + 100}}$$

1.2 Part 2: When does the hunter shoot and miss when closest to the rabbit?

To find when the hunter is closest to the rabbit, we calculate the distance between the hunter and the rabbit as a function of time. The distance function $D(t)$ is given by:

$$D(t) = \|\text{Hunter's position} - \text{Rabbit's position}\| = \|\langle 10 - t, 0\rangle - \langle t, 2t\rangle\|$$

$$D(t) = \|\langle 10 - t - t, 0 - 2t\rangle\| = \|\langle 10 - 2t, -2t\rangle\|$$
$$D(t) = \sqrt{(10 - 2t)^2 + (-2t)^2} = \sqrt{(10 - 2t)^2 + 4t^2}$$

We minimize this distance by finding the critical points of $D(t)$. We first differentiate $D(t)$ with respect to t. Let:

$$f(t) = (10 - 2t)^2 + 4t^2$$

Then:

$$f'(t) = 2(10 - 2t)(-2) + 2(4t) = -4(10 - 2t) + 8t$$
$$f'(t) = -40 + 8t + 8t = -40 + 16t$$

Setting $f'(t) = 0$ to find the critical points:

$$-40 + 16t = 0 \quad \Rightarrow \quad t = \frac{40}{16} = 2.5$$

Thus, the hunter is closest to the rabbit at time $t = 2.5$.

1.3 Conclusion

1. The vector-valued function for the arrow at time t is:

$$A(t) = \frac{\langle 2t - 10, 2t \rangle}{\sqrt{8t^2 - 40t + 100}}$$

2. The hunter is closest to the rabbit at time $t = 2.5$.

2 Solution

Let $a(t)$ and $b(t)$ be vector-valued functions in $\mathbb{R}^3$, and let $a(t) = \langle a_1(t), a_2(t), a_3(t) \rangle$ and $b(t) = \langle b_1(t), b_2(t), b_3(t) \rangle$. We will show how the derivative interacts with the dot product and cross product of these vector-valued functions.

2.1 Part 1: Derivative of the dot product

We wish to show that the derivative of the dot product of $a(t)$ and $b(t)$ satisfies:

$$\frac{d}{dt}(a(t) \cdot b(t)) = a'(t) \cdot b(t) + a(t) \cdot b'(t)$$

The dot product of $a(t)$ and $b(t)$ is:

$$a(t) \cdot b(t) = a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t)$$

Taking the derivative with respect to t:

$$\frac{d}{dt}(a(t) \cdot b(t)) = \frac{d}{dt}(a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t))$$

Applying the product rule for each term:

$$= \frac{d}{dt}(a_1(t)b_1(t)) + \frac{d}{dt}(a_2(t)b_2(t)) + \frac{d}{dt}(a_3(t)b_3(t))$$

$$= a_1'(t)b_1(t) + a_1(t)b_1'(t) + a_2'(t)b_2(t) + a_2(t)b_2'(t) + a_3'(t)b_3(t) + a_3(t)b_3'(t)$$

Rearranging the terms:

$$= \big(a_1'(t)b_1(t) + a_2'(t)b_2(t) + a_3'(t)b_3(t)\big) + \big(a_1(t)b_1'(t) + a_2(t)b_2'(t) + a_3(t)b_3'(t)\big)$$

$$= a'(t) \cdot b(t) + a(t) \cdot b'(t)$$

Thus, we have shown:

$$\frac{d}{dt}(a(t) \cdot b(t)) = a'(t) \cdot b(t) + a(t) \cdot b'(t)$$

2.2 Part 2: Derivative of the cross product

We wish to show that the derivative of the cross product of $a(t)$ and $b(t)$ satisfies:

$$\frac{d}{dt}\left(a(t) \times b(t)\right) = a'(t) \times b(t) + a(t) \times b'(t)$$

The cross product of $a(t)$ and $b(t)$ is given by:

$$a(t) \times b(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1(t) & a_2(t) & a_3(t) \\ b_1(t) & b_2(t) & b_3(t) \end{vmatrix}$$

To take the derivative of the cross product, we differentiate each component of the vector:

$$\frac{d}{dt}\left(a(t) \times b(t)\right) = \frac{d}{dt}\begin{pmatrix} a_2(t)b_3(t) - a_3(t)b_2(t) \\ a_3(t)b_1(t) - a_1(t)b_3(t) \\ a_1(t)b_2(t) - a_2(t)b_1(t) \end{pmatrix}$$

Applying the product rule to each component:

$$= \begin{pmatrix} a_2'(t)b_3(t) + a_2(t)b_3'(t) - a_3'(t)b_2(t) - a_3(t)b_2'(t) \\ a_3'(t)b_1(t) + a_3(t)b_1'(t) - a_1'(t)b_3(t) - a_1(t)b_3'(t) \\ a_1'(t)b_2(t) + a_1(t)b_2'(t) - a_2'(t)b_1(t) - a_2(t)b_1'(t) \end{pmatrix}$$

We can rearrange this as:

$$= \begin{pmatrix} a_2'(t)b_3(t) - a_3'(t)b_2(t) \\ a_3'(t)b_1(t) - a_1'(t)b_3(t) \\ a_1'(t)b_2(t) - a_2'(t)b_1(t) \end{pmatrix} + \begin{pmatrix} a_2(t)b_3'(t) - a_3(t)b_2'(t) \\ a_3(t)b_1'(t) - a_1(t)b_3'(t) \\ a_1(t)b_2'(t) - a_2(t)b_1'(t) \end{pmatrix}$$

The first term corresponds to $a'(t) \times b(t)$, and the second term corresponds to $a(t) \times b'(t)$. Thus, we have:

$$\frac{d}{dt}\left(a(t) \times b(t)\right) = a'(t) \times b(t) + a(t) \times b'(t)$$

3 Solution

The position vector of a point P is given by:

$$\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j} + 4\cos(t)\mathbf{k}$$

We will analyze this motion in the following parts.

3.1 Part 1: Show that it moves on the surface of a sphere centered at the origin

To show that the point P moves on the surface of a sphere centered at the origin, we calculate the magnitude of the position vector $\mathbf{r}(t)$.

The magnitude $|\mathbf{r}(t)|$ is given by:

$$\begin{aligned}
|\mathbf{r}(t)| &= \sqrt{(3\cos(t))^2 + (5\sin(t))^2 + (4\cos(t))^2} \\
&= \sqrt{9\cos^2(t) + 25\sin^2(t) + 16\cos^2(t)} \\
&= \sqrt{(9+16)\cos^2(t) + 25\sin^2(t)} \\
&= \sqrt{25\cos^2(t) + 25\sin^2(t)} \\
&= \sqrt{25(\cos^2(t) + \sin^2(t))} = \sqrt{25} = 5
\end{aligned}$$

Since the magnitude of the position vector is constant (5), the point P moves on the surface of a sphere with radius 5 centered at the origin.

3.2 Part 2: Show that it also moves on a plane through the origin

To show that the point P moves on a plane, we analyze the coordinates of the position vector. The position vector is:

$$\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$$

To check if P lies on a plane, we look for a relationship between the components. Notice that:

$$\frac{x(t)}{3} = \cos(t), \quad \frac{z(t)}{4} = \cos(t)$$

Thus, we can eliminate the parameter t:

$$\frac{x(t)}{3} = \frac{z(t)}{4} \quad \Rightarrow \quad 4x(t) = 3z(t)$$

This is the equation of a plane through the origin. Therefore, the point P also moves on the plane $4x - 3z = 0$.

3.3 Part 3: Show that its speed is constant

The speed of the point P is given by the magnitude of the velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$.

First, calculate the velocity vector:

$$\mathbf{v}(t) = \frac{d}{dt}\left(3\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j} + 4\cos(t)\mathbf{k}\right)$$

$$= -3\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j} - 4\sin(t)\mathbf{k}$$

Now, compute the magnitude of the velocity vector:

$$|\mathbf{v}(t)| = \sqrt{(-3\sin(t))^2 + (5\cos(t))^2 + (-4\sin(t))^2}$$

$$= \sqrt{9\sin^2(t) + 25\cos^2(t) + 16\sin^2(t)}$$

$$= \sqrt{(9+16)\sin^2(t) + 25\cos^2(t)}$$

$$= \sqrt{25\sin^2(t) + 25\cos^2(t)}$$

$$= \sqrt{25(\sin^2(t) + \cos^2(t))} = \sqrt{25} = 5$$

Since the magnitude of the velocity vector is constant (5), the speed of the point P is constant.

3.4 Part 4: Show that the acceleration is directed toward the origin

The acceleration vector is the derivative of the velocity vector:

$$\begin{aligned}\mathbf{a}(t) &= \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\left(-3\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j} - 4\sin(t)\mathbf{k}\right)\\ &= -3\cos(t)\mathbf{i} - 5\sin(t)\mathbf{j} - 4\cos(t)\mathbf{k}\end{aligned}$$

Thus, the acceleration vector is:

$$\mathbf{a}(t) = \langle -3\cos(t), -5\sin(t), -4\cos(t)\rangle$$

Notice that $\mathbf{a}(t)$ is a scalar multiple of $\mathbf{r}(t)$:

$$\mathbf{a}(t) = -1 \cdot \mathbf{r}(t)$$

Therefore, the acceleration vector is always directed toward the origin, as it is the negative of the position vector.

3.5 Conclusion

1. The point P moves on the surface of a sphere with radius 5 centered at the origin.
2. The point P moves on the plane $4x - 3z = 0$.
3. The speed of the point P is constant and equal to 5.
4. The acceleration of the point P is always directed toward the origin.