EXAM REVIEW (RECITATION 7)

1. Problems from official course

1.1. Vectors. Let $P = (1, 0, 1)$, $Q = (1, 1, 2)$, and $R = (-1, 1, 1)$.

(1) What is the vector connecting P to the *midpoint* of the line segment connecting Q and R?
(2) What is the area of the triangle with vertices P, Q, R?
(3) What is the equation of the plane through these three points?

1.2. Quadratic. Suppose we have real numbers $x_1, x_2, x_3, y_1, y_2, y_3$ satisfying the conditions $x_1^2 + x_2^2 + x_3^2 = 4$ and $y_1^2 + y_2^2 + y_3^2 = 9$. What is the range of possible values for

$$x_1 y_1 + x_2 y_2 + x_3 y_3?$$

1.3. Planes. Let P_1 be the plane with equation $x + 2y + 3z = 0$ and P_2 be the plane with equation $2y - z = 0$.

(1) Write down a vector **parallel** to both P_1 and P_2.
(2) Find the distance from the point $(2, 1, 4)$ to the plane P_1.

1.4. Matrix.

(1) Calculate the matrix M associated to the linear transformation of $\mathbf{R}^2$ given by rotation **counterclockwise** by $5\pi/4$.
(2) Let $N = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 6 & 2 \end{pmatrix}$. Calculate (if defined) the matrix products MN and NM.

1.5. Parallellopiped. Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

(1) What is the volume of the parallelopiped whose sides are given by the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?
(2) Are these vectors a basis for $\mathbf{R}^3$?

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1.6. **System.** Consider the following system of equations with unknown variables x, y, which depend on a real parameter a

$$x + 3y = 0$$
$$-ax - y = 1$$

(1) Write this system of equations as a single matrix equation with a vector unknown.
(2) Find all values of a for which this system has a unique solution.
(3) By computing the inverse of a 2×2 matrix, find the solution to this equation in terms of a.

1.7. **Eigenvectors.** Consider the matrix

$$A = \begin{pmatrix} 5 & 8 \\ 7 & 4 \end{pmatrix}.$$

(1) Calculate the characteristic polynomial and the eigenvalues of A.
(2) For the largest eigenvalue, find a corresponding eigenvector $\mathbf{v}$.

1.8. **Complex.**
(1) Calculate $(1 - i\sqrt{3})^7$.
(2) Let $z = 2 + 3i$ and $w = 1 + 2i$. Find zw, $z/\overline{w}$.

2. Older problems from Evan

(1) In $\mathbb{R}^3$, compute the projection of the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ onto the plane $x + y + 2z = 0$.
(2) Suppose A, B, C, D are points in $\mathbb{R}^3$. Give a geometric interpretation for this expression:
$$|\vec{DA} \cdot \left(\vec{DB} \times \vec{DC}\right)|.$$
(3) Fix a plane $\mathcal{P}$ in $\mathbb{R}^3$ which passes through the origin. Consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(\mathrm{v})$ is the projection of v onto $\mathcal{P}$. Let M denote the 3×3 matrix associated to f. Compute the determinant of M.
(4) Let a and b be two perpendicular unit vectors in $\mathbb{R}^3$. A third vector v in $\mathbb{R}^3$ lies in the span of a and b. Given that $\mathrm{v} \cdot \mathrm{a} = 2$ and $\mathrm{v} \cdot \mathrm{b} = 3$, compute the magnitudes of the cross products $\mathrm{v} \times \mathrm{a}$ and $\mathrm{v} \times \mathrm{b}$.
(5) Compute the trace of the 2×2 matrix M given the two equations
$$M \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \text{ and } M \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$
(6) There are three complex numbers z satisfying $z^3 = 5 + 6i$. Suppose we plot these three numbers in the complex plane. Compute the area of the triangle they enclose.