Quick answer key to Recitation 6

ChatGPT 4o

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Use the table of contents below to skip to a specific part without seeing spoilers to the other parts.

I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution **2**

1.1 Part 1: Convert $(x, y) = (-\sqrt{3}, 1)$ to polar coordinates 2

1.2 Part 2: Convert $(r, \theta) = (3, \pi/6)$ to Cartesian coordinates . . 3

1.3 Part 3: Convert $(x, y) = (-\sqrt{6}, -\sqrt{2})$ to polar coordinates . . 4

2 Solution **5**

2.1 Part 1: Show that $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ 5

2.2 Part 2: Show that $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ 6

2.3 Part 3: Express $\sin^3(\theta)$ in terms of $\sin(3\theta)$ and $\sin(\theta)$ 7

3 Solution **8**

3.1 Part 1: Find the real and imaginary parts of $f(t)$ 8

3.2 Part 2: Find $\overline{f(t)}$ and $|f(t)|^2$ 9

4 Solution **10**

4.1 Part 1: Fourth power of $2 + 2i$ 10

4.2 Part 2: Fourth power of $-3 + i\sqrt{3}$ 11

4.3 Part 3: Graphing the numbers and their fourth powers 12

5 Solution **13**

5.1 Step 1: Find the characteristic polynomial 14

5.2 Step 2: Find the eigenvectors 15

5.3 Conclusion 17

1 Solution

We are asked to convert the following points between Cartesian and polar coordinates.

1.1 Part 1: Convert $(x, y) = (-\sqrt{3}, 1)$ to polar coordinates

Given $(x, y) = (-\sqrt{3}, 1)$, we convert to polar coordinates using the formulas:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

First, we compute the modulus r:

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2$$

Next, we compute the argument θ:

$$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$

Since $x = -\sqrt{3}$ and $y = 1$, the point lies in the second quadrant. We first compute the reference angle:

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Thus, the argument is:

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Therefore, the polar coordinates are:

$$(r, \theta) = (2, \frac{5\pi}{6})$$

1.2 Part 2: Convert $(r, \theta) = (3, \pi/6)$ to Cartesian coordinates

Given $(r, \theta) = (3, \pi/6)$, we convert to Cartesian coordinates using the formulas:

$$x = r\cos\theta, \quad y = r\sin\theta$$

First, we compute x:

$$x = 3\cos\left(\frac{\pi}{6}\right) = 3 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Next, we compute y:

$$y = 3\sin\left(\frac{\pi}{6}\right) = 3 \times \frac{1}{2} = \frac{3}{2}$$

Thus, the Cartesian coordinates are:

$$(x, y) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

1.3 Part 3: Convert $(x, y) = (-\sqrt{6}, -\sqrt{2})$ to polar coordinates

Given $(x, y) = (-\sqrt{6}, -\sqrt{2})$, we use the same formulas as in Part 1.

First, we compute r:

$$r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2}$$

Next, we compute θ:

$$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{6}}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

The reference angle is $\frac{\pi}{6}$. Since both x and y are negative, the point lies in the third quadrant, so:

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Therefore, the polar coordinates are:

$$(r, \theta) = \left(2\sqrt{2}, \frac{7\pi}{6}\right)$$

2 Solution

2.1 Part 1: Show that $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

We start with Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Subtract the second equation from the first:

$$e^{i\theta} - e^{-i\theta} = (\cos(\theta) + i\sin(\theta)) - (\cos(\theta) - i\sin(\theta)) = 2i\sin(\theta)$$

Solving for $\sin(\theta)$, we get:

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Thus, we have shown that:

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

2.2 Part 2: Show that $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

Starting again with Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Add the two equations:

$$e^{i\theta} + e^{-i\theta} = (\cos(\theta) + i\sin(\theta)) + (\cos(\theta) - i\sin(\theta)) = 2\cos(\theta)$$

Solving for $\cos(\theta)$, we get:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Thus, we have shown that:

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

2.3 Part 3: Express $\sin^3(\theta)$ in terms of $\sin(3\theta)$ and $\sin(\theta)$

We can express $\sin^3(\theta)$ using the exponential form of $\sin(\theta)$:

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Thus:

$$\sin^3(\theta) = \left(\frac{1}{2i}(e^{i\theta} - e^{-i\theta})\right)^3 = \frac{1}{(2i)^3}(e^{i\theta} - e^{-i\theta})^3$$

Now, expand $(e^{i\theta} - e^{-i\theta})^3$ using the binomial theorem:

$$(e^{i\theta} - e^{-i\theta})^3 = e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}$$

Thus:

$$\sin^3(\theta) = \frac{1}{-8i}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta})$$

We separate this expression into two parts:

$$\sin^3(\theta) = \frac{1}{8}\left(\frac{1}{i}(e^{3i\theta} - e^{-3i\theta})\right) - \frac{3}{8i}(e^{i\theta} - e^{-i\theta})$$

Using the identity for $\sin(\theta)$, we can express this as:

$$\sin^3(\theta) = \frac{1}{4}\sin(3\theta) - \frac{3}{4}\sin(\theta)$$

Thus, we have:

$$\sin^3(\theta) = \frac{1}{4}\sin(3\theta) - \frac{3}{4}\sin(\theta)$$

3 Solution

We are given the complex function:

$$f(t) = \frac{t+2i}{1-3i}$$

where t is a real number. We are asked to find the real and imaginary parts of $f(t)$, as well as $\overline{f(t)}$ and $|f(t)|^2$.

3.1 Part 1: Find the real and imaginary parts of $f(t)$

To find the real and imaginary parts of $f(t)$, we first simplify the expression by multiplying the numerator and denominator by the complex conjugate of the denominator $1-3i$, which is $1+3i$:

$$f(t) = \frac{t+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(t+2i)(1+3i)}{(1-3i)(1+3i)}$$

First, simplify the denominator:

$$(1-3i)(1+3i) = 1^2 - (3i)^2 = 1 - (-9) = 1 + 9 = 10$$

Next, expand the numerator:

$$(t+2i)(1+3i) = t(1+3i) + 2i(1+3i) = t + 3ti + 2i + 6i^2$$

Since $i^2 = -1$, this becomes:

$$t + 3ti + 2i - 6 = (t-6) + (3t+2)i$$

Thus, we have:

$$f(t) = \frac{(t-6)+(3t+2)i}{10}$$

We can now separate the real and imaginary parts:

$$f(t) = \frac{t-6}{10} + \frac{3t+2}{10}i$$

Therefore, the real and imaginary parts of $f(t)$ are:

$$\text{Re}(f(t)) = \frac{t-6}{10}, \quad \text{Im}(f(t)) = \frac{3t+2}{10}$$

3.2 Part 2: Find $\overline{f(t)}$ and $|f(t)|^2$

The complex conjugate $\overline{f(t)}$ is obtained by changing the sign of the imaginary part:

$$\overline{f(t)} = \frac{t-6}{10} - \frac{3t+2}{10}i$$

Next, we compute $|f(t)|^2$, which is given by:

$$|f(t)|^2 = f(t) \cdot \overline{f(t)} = \left(\frac{t-6}{10} + \frac{3t+2}{10}i\right)\left(\frac{t-6}{10} - \frac{3t+2}{10}i\right)$$

Using the identity $(a+bi)(a-bi) = a^2 + b^2$, we get:

$$|f(t)|^2 = \left(\frac{t-6}{10}\right)^2 + \left(\frac{3t+2}{10}\right)^2$$

$$= \frac{(t-6)^2}{100} + \frac{(3t+2)^2}{100}$$

$$= \frac{(t-6)^2 + (3t+2)^2}{100}$$

Now, expand the terms:

$$(t-6)^2 = t^2 - 12t + 36, \quad (3t+2)^2 = 9t^2 + 12t + 4$$

Adding them together:

$$(t-6)^2 + (3t+2)^2 = t^2 - 12t + 36 + 9t^2 + 12t + 4 = 10t^2 + 40$$

Thus:

$$|f(t)|^2 = \frac{10t^2 + 40}{100} = \frac{t^2 + 4}{10}$$

4 Solution

We are tasked with finding the fourth powers of $2+2i$ and $-3+i\sqrt{3}$ using their polar forms. Afterward, we will graph these numbers and their fourth powers on the complex plane.

4.1 Part 1: Fourth power of $2+2i$

First, we express $2+2i$ in polar form. The modulus r of $2+2i$ is:

$$r = |2+2i| = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

Next, we compute the argument θ:

$$\theta = \arg(2+2i) = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

Thus, the polar form of $2+2i$ is:

$$2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

To find the fourth power, we use De Moivre's theorem:

$$\begin{aligned}(2+2i)^4 &= \left(2\sqrt{2}\right)^4\left(\cos\left(4\times\frac{\pi}{4}\right) + i\sin\left(4\times\frac{\pi}{4}\right)\right)\\ &= (2\sqrt{2})^4\,(\cos\pi + i\sin\pi)\\ &= 64(-1) = -64\end{aligned}$$

Thus, the fourth power of $2+2i$ is -64.

4.2 Part 2: Fourth power of $-3 + i\sqrt{3}$

Next, we express $-3 + i\sqrt{3}$ in polar form. The modulus r is:

$$r = |-3 + i\sqrt{3}| = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

The argument θ is:

$$\theta = \arg(-3 + i\sqrt{3}) = \pi - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Thus, the polar form of $-3 + i\sqrt{3}$ is:

$$-3 + i\sqrt{3} = 2\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Now, we find the fourth power using De Moivre's theorem:

$$(-3 + i\sqrt{3})^4 = \left(2\sqrt{3}\right)^4 \left(\cos\left(4 \times \frac{5\pi}{6}\right) + i\sin\left(4 \times \frac{5\pi}{6}\right)\right)$$

$$= (2\sqrt{3})^4 \left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$$

Since $\frac{10\pi}{3} = 2\pi + \frac{4\pi}{3}$, we simplify to:

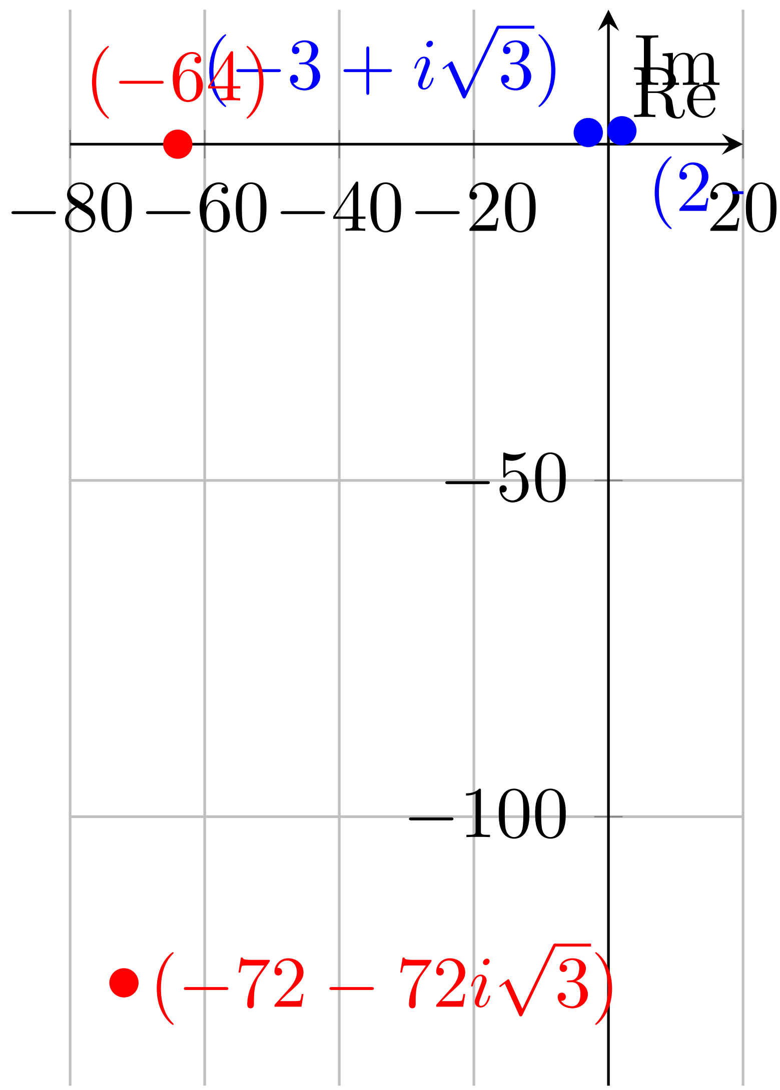
$$(-3 + i\sqrt{3})^4 = 144\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 144\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -72 - 72i\sqrt{3}$$

Thus, the fourth power of $-3 + i\sqrt{3}$ is $-72 - 72i\sqrt{3}$.

4.3 Part 3: Graphing the numbers and their fourth powers

The following plot shows the numbers $2+2i$ and $-3+i\sqrt{3}$, along with their fourth powers, -64 and $-72-72i\sqrt{3}$, respectively.



5 Solution

We are tasked with finding the complex eigenvalues and eigenvectors of the matrix:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

5.1 Step 1: Find the characteristic polynomial

The eigenvalues are solutions to the characteristic equation:

$$\det(A - \lambda I) = 0$$

where I is the identity matrix and λ is the eigenvalue. First, compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

Now, compute the determinant of this matrix:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)(-\lambda) - (-1)(1)$$

$$= \lambda^2 - 1$$

Thus, the characteristic equation is:

$$\lambda^2 + 1 = 0$$

Solving for λ, we get:

$$\lambda^2 = -1 \quad \Rightarrow \quad \lambda = \pm i$$

Therefore, the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

5.2 Step 2: Find the eigenvectors

For each eigenvalue, we solve the system $(A - \lambda I)\mathbf{v} = 0$ where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is the eigenvector.

Eigenvalue $\lambda_1 = i$: We solve:

$$(A - iI)\mathbf{v} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$-iv_1 - v_2 = 0 \quad (1)$$

$$v_1 - iv_2 = 0 \quad (2)$$

From equation (2), solve for v_1:

$$v_1 = iv_2$$

Substitute $v_1 = iv_2$ into equation (1):

$$-i(iv_2) - v_2 = 0 \quad \Rightarrow \quad v_2 + v_2 = 0 \quad \Rightarrow \quad v_2 = 0$$

If $v_2 = 0$, then from equation (2), $v_1 = 0$, but this would not provide a valid eigenvector. Thus, assume $v_2 = 1$, which implies $v_1 = i$.

Therefore, the eigenvector corresponding to $\lambda_1 = i$ is:

$$\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = -i$: We solve:

$$(A + iI)\mathbf{v} = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$iv_1 - v_2 = 0 \quad (3)$$

$$v_1 + iv_2 = 0 \quad (4)$$

From equation (4), solve for v_1:

$$v_1 = -iv_2$$

Substitute $v_1 = -iv_2$ into equation (3):

$$i(-iv_2) - v_2 = 0 \quad \Rightarrow \quad v_2 + v_2 = 0 \quad \Rightarrow \quad v_2 = 0$$

If $v_2 = 0$, this would lead to $v_1 = 0$, which is not a valid eigenvector. Therefore, assume $v_2 = 1$, which gives $v_1 = -i$.

Thus, the eigenvector corresponding to $\lambda_2 = -i$ is:

$$\mathbf{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

5.3 Conclusion

The eigenvalues of the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are:

$$\lambda_1 = i, \quad \lambda_2 = -i$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$