# Notes for 18.02 Recitation 6

## 18.02 Recitation MW9

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*Run, you clever boy, and remember.*

— *Clara Oswald, in Doctor Who*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

- Optional midterm review, 4-270, Thu 4:30pm-6:30pm. Led by me, Vinjay, and Sebastian.
- Please fill out the survey at https://forms.gle/AsXPweCbJ1Nvzp3k7 when you can.
- I made a Discord server. If you didn't get the link emailed to you, ask me to join.
- Remember that September 30 is the last day to switch sections freely on Canvas.

## §1 It's a miracle that multiplication in $\mathbb{C}$ has geometric meaning

Let $\mathbb{C}$ denote the set of complex numbers (just as $\mathbb{R}$ denotes the real numbers). It's important that realize that, **until we add in complex multiplication, $\mathbb{C}$ is just an elaborate $\mathbb{R}^2$ cosplay**.

| Concept | For $\mathbb{R}^2$ | For $\mathbb{C}$ |
|---|---|---|
| Notation | $\mathbf{v}$ | $z$ |
| Components | $\binom{x}{y}$ | $x+yi$ |
| Length | Length $\lvert\mathbf{v}\rvert$ | Abs val $\lvert z\rvert$ |
| Direction | (slope, maybe?) | argument $\theta$ |
| Length 1 | unit vector | $e^{i\theta}=\cos\theta+i\sin\theta$ |
| Multiply | *NONE* | ✨ $z_1z_2$ ✨ |

- All the way back in R01, when I warned you about type safety, I repeatedly stressed you that you **cannot multiply two vectors in $\mathbb{R}^n$ to get another vector**. You had a "dot product", but it spits out a number. (Honestly, you shouldn't think of dot product as a "product"; the name sucks.)
- Of course, the classic newbie mistake (which you better not make on your midterm) is to define a product on vectors component-wise: why can't $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ have "product" $\begin{pmatrix} x_1y_1 \\ \vdots \\ x_ny_n \end{pmatrix}$? Well, in 18.02, every vector definition needed a corresponding geometric picture for us to consider it worthy of attention (see table at start of `r03.pdf`). This definition has no geometric meaning.
- However, there is a big miracle for $\mathbb{C}$. For complex numbers, you can define multiplication by $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$ and there is *an amazing geometric interpretation.*

Unfortunately, AFAIK there is no English word for "complex number whose absolute value is one" (err, CNWAVIO?), the same way there is for "unit vector". For 18.02, we instead use $e^{i\theta}:=\cos\theta+i\sin\theta$ as the "word"; whenever you see $e^{i\theta}$, draw it as unit vector $\cos\theta+i\sin\theta$.

It's worth pointing out the notation $e^{i\theta}$ should strike you as *nonsense*. What meaning does it have to raise a number to an imaginary power? Does $i^i$ have a meaning? Does $\cos(i)$ have a meaning? (If you want to know, check Section 4.1 in the post-recitation notes.)

But for 18.02, when starting out, I would actually think of the notation $e^{i\theta}$ as a *mnemonic*, i.e. a silly way to remember the following result:

**Theorem 1.1**: If you multiply two CNWAVIO's, you get the CNWAVIO with the angles added:

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \Longleftrightarrow \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2).$$

More generally, multiplying two complex numbers multiplies the norms and adds the angles.

This is IMO the biggest miracle in all of precalculus. Corollary: $e^{in\theta} = \left(e^{i\theta}\right)^n \Longleftrightarrow (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, allows taking $n$th roots; Maulik showed $z^2 = 2i$ in class.

## §2 Rectangular vs polar

Every complex number can be written in either *rectangular form* ($a + bi$ for $a, b \in \mathbb{R}$) or *polar form* ($re^{i\theta}$). Depending on what you are trying to do, some forms are easier to work with than others.

| Operation | In rectangular | In polar |
|---|---|---|
| $z_1 \pm z_2$ | ✓ Component-wise like in $\mathbb{R}^2$ | ✗ Unless $z_1$ is a real multiple of $z_2$ |
| $z_1 z_2$ | ✓ Expanding | ✓ by Theorem 1.1 |
| $z_1/z_2$ | ✓ Use $\frac{1}{c+di} = \frac{c-di}{c^2+d^2}$ trick then multiply | ✓ by Theorem 1.1 |
| $n^{\text{th}}$ root of $z_1$ | ✗ Not recommended for $n > 1$ | ✓ by Theorem 1.1 corollary |

## §3 Recitation problems from official course

**1.** For each of the following points, convert it from Cartesian to polar or vice versa:
- $(x, y) = \left(-\sqrt{3}, 1\right)$
- $(r, \theta) = (3, \pi/6)$
- $(x, y) = \left(-\sqrt{6}, -\sqrt{2}\right)$

**2.** Show that $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ and $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. Use this to write $(\sin(\theta))^3$ in terms $\sin(3\theta)$ and $\sin(\theta)$.

**3.** Consider the complex number $f(t) = \frac{t+2i}{1-3i}$ where $t$ is real.
- Find the real and imaginary part of $f(t)$.
- Find $\overline{f(t)}$ and $|f(t)|^2$.

**4.** Use polar form to find the fourth powers of $2 + 2i$ and $-3 + i\sqrt{3}$. Graph these numbers and their fourth powers on the complex plane.

**5.** (If you have time) Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In class, working with real numbers, this had no eigenvectors. But now we can treat it as a matrix with complex number entries. Find complex number eigenvalues for $A$ and for each one, find an eigenvector in $\mathbb{C}^2$, i.e. a two-component vector $\binom{z}{w}$ where $z, w$ are complex numbers.

# §4 Post-recitation notes

## §4.1 The importance of definitions; also $\cos(i)$ and $i^i$ (not for exam)

When learning mathematics, I believe definitions are actually more important than theorems. A lot of confusion comes from not having been given careful definitions of the objects. (See `https://web.evanchen.cc/handouts/NaturalProof/NaturalProof.pdf` for more on that.)

So in general any time you are confused about whether an operation is "legal" — and this is true in all of math, not just 18.02 — **the first thing to really check whether you have been given a precise definition**. The endless Internet debates on whether 0 is even or whether $0.999... = 1$ or whether $\frac{1}{x}$ is a continuous function (hint: yes) are all examples of people who don't know the definitions of objects they're dsicussing.

### §4.1.1 Real exponents, real base

With that in mind, let's fix $a > 0$ a positive real number and think about what $a^r$ should mean.

**Definition 4.1** (18.100 definition):

- When $n > 0$ is an integer, then $a^n := a \times \ldots \times a$, where $a$ is repeated $n$ times.
- Then we let $a^{-n} := \frac{1}{a^n}$ for each integer $n > 0$.
- When $\frac{m}{n}$ is a rational number, $a^{\frac{m}{n}}$ means the unique $b > 0$ such that $a^m = b^n$. (In 18.100, one proves this $b$ is unique and does exist.)
- It's less clear what $a^x$ means when $x \in \mathbb{R}$, like $x = \sqrt{2}$ or $x = \pi$. I think usually one takes a limit of rational numbers $q$ close to $x$ and lets $a^x := \lim_{q \to x} a^q$. (In 18.100, one proves this limit does in fact exist.)

### §4.1.2 Complex exponents, real base

But when $z \in \mathbb{C}$, what does $a^z$ mean? There's no good way to do this.

You likely don't find an answer until 18.112, but I'll tell you now. In 18.100 you will also prove that the Taylor series

$$e^x = \sum_{k \geq 0} \frac{r^k}{k!}$$

is correct, where $e := \sum_{k \geq 0} \frac{1}{k!}$ is Euler's constant.

So then when you start 18.112, we will flip the definition on its head:

**Definition 4.2** (18.112 definition): If $z \in \mathbb{C}$, we *define*

$$e^z := \sum_{k \geq 0} \frac{z^k}{k!}.$$

Then for $a > 0$, we let $a^z = e^{z \log a}$.

To summarize: in 18.100, we defined exponents in the way you learned in grade school and then proved there was a Taylor series. But in 18.112, you *start* with the Taylor series and *then* prove that the rules in grade school you learned still applied.

And checking this consistency requires work. Because we threw away Definition 4.1, identities like $e^{z_1+z_2} = e^{z_1}e^{z_2}$ and $(e^{z_1})^{z_2} = e^{z_1 z_2}$ are no longer "free": they have to be proved rigorously too. (To be fair, they need to be proved in 18.100 too, but there it's comparatively easier.) I think you shouldn't be *surprised* they're true; we know it's true for $\mathbb{R}$, so it's one heck of a good guess. But you shouldn't take these on faith. At least get your professor to acknowledge they *require* a (non-obvious) proof, even if you aren't experienced enough to follow the proof yourself yet.

Anyway, if we accept this definition, then Euler's formula makes more sense:

**Theorem 4.3** (Euler): We have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The point is that cosine and sine also have a Taylor series that is compatible with definition:

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots. \end{aligned} \tag{1}$$

And if you put these together, you can verify Theorem 4.3, up to some technical issues with infinite sums. I think Maulik even showed this in class:

$$\begin{aligned} \cos(\theta) + i\sin(\theta) &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)i \\ &= 1 + (\theta i) + \frac{(\theta i)^2}{2!} + \frac{(\theta i)^3}{3!} + \frac{\theta i^4}{4!} + \frac{(\theta i)^5}{5!} \\ &= e^{i\theta}. \end{aligned}$$

### §4.1.3 Complex exponents, complex base

But what about $i^i$? Our Definition 4.2 above only worked for positive real numbers $a > 0$. Here, it turns out you're out of luck. There isn't any way to define $i^i$ in a way that makes internal sense. The problem is that there's no way to take a single log of a complex number, so the analogy with $\log a$ breaks down.

Put another way: there's no good way to assign a value to $\log(i)$, because $e^{i\pi/2} = e^{5i\pi/2} = \dots$ are all equal to $i$. You might hear this phrased "complex-valued logarithms are multivalued". You can have some fun with this paradox:

$$\begin{aligned} i = e^{i\pi/2} &\Longrightarrow i^i = e^{-\pi/2} \\ i = e^{5i\pi/2} &\Longrightarrow i^i = e^{-5\pi/2}. \end{aligned}$$

Yeah, trouble.

### §4.1.4 Trig functions with complex arguments

On the other hand, $\cos(i)$ can be defined: use the Taylor series Equation 1, like we did for $e^z$. To spell it out:

**Definition 4.4** (18.112 trig definitions): If $z$ is a complex number, we define

$$\begin{aligned} \cos(z) &:= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \\ \sin(z) &:= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots. \end{aligned}$$

If you do this, then Definition 4.2 implies the following identities are kosher:

**Proposition 4.5**: Under Definition 4.4, we have the identities

$$\begin{aligned} \cos(z) &:= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &:= \frac{e^{iz} - e^{-iz}}{2i}. \end{aligned}$$

*Proof*: If you write out $e^{iz} = \sum \frac{(iz)^k}{k!}$ and $e^{-iz} = \sum \frac{(-iz)^k}{k!}$ and add them, the odd $k$'s cancel out and the even $k$'s don't, which gives you

$$e^{iz} + e^{-iz} = 2 - 2 \cdot \frac{z^2}{2!} + 2 \cdot \frac{z^4}{4!} - 2 \cdot \frac{z^6}{6!} + \ldots.$$

So dividing by 2, we see $\cos(z)$ on the right-hand side, as needed. The argument with sin is similar, but this time the even $k$'s cancel and you divide by $2i$ instead. □

So for example, from Proposition 4.5, we conclude for example that

$$\cos(i) = \frac{e + \frac{1}{e}}{2}.$$

Strange but true.

## §4.2 The future: what are 18.100 and 18.112 anyway? (not for exam)

First I need to tell you what analysis is. When students in USA ask me what analysis is, I sometimes say "calculus but you actually prove things". But that's actually a bit backwards; it turns out that in much parts of the world, there is no topic called "calculus".[1] It would be more accurate to say calculus is analysis with proofs, theorems, and coherent theorem statements deleted, and it only exists in some parts of the world (which is why mathematicians will tend to look down on it).

With that out of the way,

- 18.100 is real analysis, i.e. analysis of functions over $\mathbb{R}$
- 18.112 is complex analysis, i.e. analysis of functions over $\mathbb{C}$.

[1] See `https://web.evanchen.cc/faq-school.html#S-10`.

If you ever take either class, I think the thing to know about them is:

> *Complex analysis is the good twin and real analysis is the evil one: beautiful formulas and elegant theorems seem to blossom spontaneously in the complex domain, while toil and pathology rule the reals*
>
> — *Charles Pugh, in* Real Mathematical Analysis

Personally, I think it's insane that MIT uses 18.100 as their "intro to proofs" topic. (This is why 18.100 is a prerequisite for 18.701, abstract algebra, which makes no sense either.)

# §5 Exponentiation (for exam)

This section is dedicated to $z^n$ and is **on-syllabus for exam**. Specifically, you ought to be able to solve equations like $z^5 = 243i$. This section shows you how.

In this whole section, you always prefer to work in polar form. So if you get input in rectangular form, you should first convert to rectangular form. Conversely, if the answer is asked for in rectangular form, you should work with polar form anyway, and only convert to rectangular output at the end.

## §5.1 Raising to the $n$th power

Before being able to extract $n$th roots, I need to make sure you know how to do $n$th powers. This is easy:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta).$$

For example,

$$(3(\cos 18^\circ + i\sin 18^\circ))^5 = 243(\cos 90^\circ + i\sin 90^\circ) = 243i.$$

## §5.2 Extracting $n$th roots

If you can run the process in forwards, then you should be able to run the process backwards too. First, I will tell you what the answer looks like:

**Theorem 5.1**: Consider solving the equation $z^n = w$, where $w$ is a given nonzero complex number, for $z$. Then you should always output exactly $n$ answers. Those $n$ answers all have magnitude $|w|^{\frac{1}{n}}$ and arguments spaced apart by $\frac{360^\circ}{n}$.

I think it's most illustrative if I show you the five answers to

$$z^5 = 243i$$

to start. Again, first we want to convert everything to polar coordinates:

$$z^5 = 243i = 243(\cos 90^\circ + i\sin 90^\circ).$$

At this point, we know that if $|z^5| = 243$, then $|z| = 3$; all the answers should have absolute 3. So the idea is to find the angles. Here are the five answers:

$$\begin{aligned}
z_1 &= 3(\cos 18^\circ + i \sin 18^\circ) \implies (z_1)^5 = 243(\cos 90^\circ + i \sin 90^\circ) \\
z_2 &= 3(\cos 90^\circ + i \sin 90^\circ) \implies (z_2)^5 = 243(\cos 450^\circ + i \sin 450^\circ) \\
z_3 &= 3(\cos 162^\circ + i \sin 162^\circ) \implies (z_3)^5 = 243(\cos 810^\circ + i \sin 810^\circ) \\
z_4 &= 3(\cos 234^\circ + i \sin 234^\circ) \implies (z_4)^5 = 243(\cos 1170^\circ + i \sin 1170^\circ) \\
z_5 &= 3(\cos 306^\circ + i \sin 306^\circ) \implies (z_5)^5 = 243(\cos 1530^\circ + i \sin 1530^\circ).
\end{aligned}$$

Here's a picture of the five numbers:

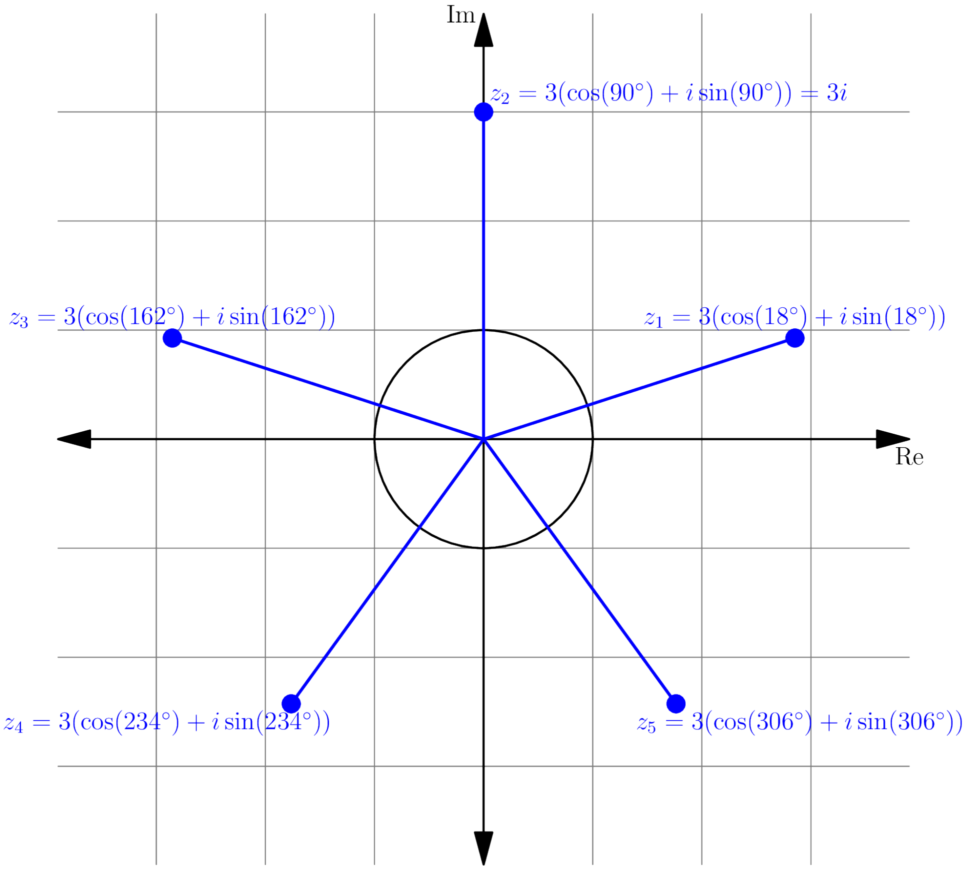


Figure 1: The five answers to $z^5 = 243i$, each of length 3.

On the right column, all the numbers are equal. Notice something interesting happening on the right-hand side. The numbers $\cos 90^\circ + i \sin 90^\circ$ and $\cos 450^\circ + i \sin 450^\circ$, etc. are all the same number; if you draw them in the plane, they'll point to the same thing. However, they give five *different* answers on the left. But if you continue the pattern one more, you start getting a cycle

$$z_6 = 3(\cos 378^\circ + i \sin 378^\circ) \implies (z_6)^5 = 243(\cos 1890^\circ + i \sin 1890^\circ).$$

This doesn't give you a new answer, because $z_6 = z_1$.

In general, if $w$ has argument $\theta$, then the arguments of $z$ satisfying $z^n = w$ start at $\frac{\theta}{n}$ and then go up in increments of $\frac{360^\circ}{n}$. (For example, they started at $\frac{90^\circ}{5} = 18^\circ$ for answers to $z^5 = 243i$.) So you can describe the general recipe as:

**Recipe for $n$th roots**

1. Convert $w$ to polar form; say it has angle $\theta$.
2. One of the $n$ answers will be $|w|^{\frac{1}{n}} \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$.
3. The other $n-1$ answers are obtained by increasing the angle in increments of $\frac{360°}{n}$.

- **Example 1**: Solve $z^5 = 243i$.

  **Answer 1**: we first convert to polar form as

  $$243i = 243(\cos 90° + i\sin 90°)$$

  and see that $243^{\frac{1}{5}} = 3$, and $\theta = 90°$. The first angle is $\frac{\theta}{5} = 18°$. So the five answers are

  $$\begin{aligned} z_1 &= 3(\cos 18° + i\sin 18°) \\ z_2 &= 3(\cos 90° + i\sin 90°) \\ z_3 &= 3(\cos 162° + i\sin 162°) \\ z_4 &= 3(\cos 234° + i\sin 234°) \\ z_5 &= 3(\cos 306° + i\sin 306°). \end{aligned}$$

  (As it happens, $z_2 = 3i$, which is easy to check by hand works.)

- **Example 2**: Solve $z^4 = 8 + 8\sqrt{3}i$.

  **Answer 2**: We first convert to polar form as

  $$8 + 8\sqrt{3}i = 16(\cos 60° + i\sin 60°)$$

  and see that $16^{\frac{1}{4}} = 2$, and $\theta = 60°$. The first angle is $\frac{\theta}{4} = 15°$. So the four answers are

  $$\begin{aligned} z_1 &= 2(\cos 15° + i\sin 15°) \\ z_2 &= 2(\cos 105° + i\sin 105°) \\ z_3 &= 2(\cos 195° + i\sin 195°) \\ z_4 &= 2(\cos 285° + i\sin 285°). \end{aligned}$$

- **Example 3**: Solve $z^3 = -1000$.

  **Answer 3**: We first convert to polar form as

  $$-1000 = 1000(\cos 180° + i\sin 180°)$$

  and see that $1000^{\frac{1}{3}} = 10$, and $\theta = 180°$. The first angle is $\frac{\theta}{3} = 60°$. So the three answers are

  $$\begin{aligned} z_1 &= 10(\cos 60° + i\sin 60°) \\ z_2 &= 10(\cos 180° + i\sin 180°) \\ z_3 &= 10(\cos 300° + i\sin 300°). \end{aligned}$$

  (As it happens, $z_2 = -10$, as expected, since $(-10)^3 = -1000$.)