Quick answer key to R03

ChatGPT 4o

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

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1 Problem 1

1.1 Part (a)

Calculate AB and BA.

First, we compute AB:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2(-3) + 1(2) & 2(1) + 1(0) \\ 1(-3) + 2(2) & 1(1) + 2(0) \end{pmatrix} = \begin{pmatrix} -6 + 2 & 2 \\ -3 + 4 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 1 & 1 \end{pmatrix}.$$

Now, we compute BA:

$$BA = \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -3(2) + 1(1) & -3(1) + 1(2) \\ 2(2) + 0(1) & 2(1) + 0(2) \end{pmatrix} = \begin{pmatrix} -6 + 1 & -3 + 2 \\ 4 + 0 & 2 + 0 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 4 & 2 \end{pmatrix}.$$

1.2 Part (b)

Calculate the matrix A^{-1}.

To calculate the inverse of A, we use the formula for a 2×2 matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

First, compute the determinant of A:

$$\det(A) = 2(2) - 1(1) = 4 - 1 = 3.$$

Now, the inverse of A is:

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

1.3 Part (c)

Check directly that $AA^{-1} = A^{-1}A = I$.

We compute AA^{-1}:

$$\begin{aligned}
AA^{-1} &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \\
&= \begin{pmatrix} 2\left(\frac{2}{3}\right) + 1\left(-\frac{1}{3}\right) & 2\left(-\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) \\ 1\left(\frac{2}{3}\right) + 2\left(-\frac{1}{3}\right) & 1\left(-\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{4}{3} - \frac{1}{3} & -\frac{2}{3} + \frac{2}{3} \\ \frac{2}{3} - \frac{2}{3} & -\frac{1}{3} + \frac{4}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.
\end{aligned}$$

Similarly, $A^{-1}A = I$.

1.4 Part (d)

Solve the system of equations using A^{-1}.

The system of equations is:

$$2x + y = 7, \quad x + 2y = 11.$$

This can be written as the matrix equation:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}.$$

Multiplying both sides by A^{-1}:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 7 \\ 11 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(7) + -\frac{1}{3}(11) \\ -\frac{1}{3}(7) + \frac{2}{3}(11) \end{pmatrix} = \begin{pmatrix} \frac{14}{3} - \frac{11}{3} \\ -\frac{7}{3} + \frac{22}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

So, the solution is $x = 1$, $y = 5$.

Check the solution.

Substituting $x = 1$ and $y = 5$ into the original equations:

$$2(1) + 5 = 7, \quad 1 + 2(5) = 11,$$

both of which are true, so the solution is correct.

2 Problem 2

2.1 (a) Transformation of the unit square by $T = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

We apply T to each vertex of the unit square:

$$T(0,0) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$T(0,1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

$$T(1,0) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

$$T(1,1) = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

The new vertices of the transformed square are $(0,0)$, $(3,1)$, $(2,0)$, and $(5,1)$.

2.2 (b) Reflection Across the Line $y = -x$

The matrix A associated with reflection across the line $y = -x$ is found by observing how the reflection changes the coordinates of a point (x, y) to $(-y, -x)$.

For the standard basis vectors:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ reflects to } \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ reflects to } \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Thus, the matrix representing this reflection is:

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

2.3 (c) Linear Transformation in $\mathbb{R}^3$

The linear transformation f acts on vectors $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ by:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2z \\ x + y + z \\ 4y + z \end{pmatrix}.$$

To find the matrix associated with this transformation, we determine how the transformation acts on the standard basis vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$:

$$f(\mathbf{e}_1) = f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad f(\mathbf{e}_2) = f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad f(\mathbf{e}_3) = f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, the matrix associated with the transformation is:

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 1 & 1 & 1 \\ 0 & 4 & 1 \end{pmatrix}.$$

2.4 (d) Projection onto the Vector $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The answer is

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$

This follows by showing $\mathbf{e}_1$ and $\mathbf{e}_2$ project to $\begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix}$ and $\begin{pmatrix} 2/5 \\ 4/5 \end{pmatrix}$ respectively.[1] Recall from R02 the projection of a vector $\mathbf{v} \in \mathbb{R}^2$ onto the vector $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is given by the formula:

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}.$$

[1] ChatGPT provided a solution by citing a formula that I don't think was covered in the class, that's basically a cheat code in my opinion. But here it is if you want to see it. The projection matrix formula for projecting onto $\mathbf{w}$ is:

$$A = \frac{1}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}\mathbf{w}^T = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Carrying out the matrix multiplication:

$$A = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$

Thus, the matrix for the projection onto $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is:

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$