Quick answer key to R02

ChatGPT 4o

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I just used ChatGPT to write this one quickly. ChatGPT can make mistakes, so if you spot anything that's wrong, flag me to ask.

Contents

1 Solution to Problem 1

Given the vectors $\mathbf{a} = \langle 2, 3, 6 \rangle$ and $\mathbf{b} = \langle 1, 2, 2 \rangle$:

1.1 Scalar Component of a in the Direction of b

The scalar component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is given by the formula:

$$\text{Scalar component} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

First, we compute the dot product $\mathbf{a} \cdot \mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{b} = 2(1) + 3(2) + 6(2) = 2 + 6 + 12 = 20$$

Next, we compute the magnitude of $\mathbf{b}$:

$$\|\mathbf{b}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

Thus, the scalar component of $\mathbf{a}$ in the direction of $\mathbf{b}$ is:

$$\frac{20}{3}$$

1.2 Vector Component of a in the Direction of b

The vector component of **a** in the direction of **b** is given by the formula:

$$\text{Vector component} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}$$

We already have $\mathbf{a} \cdot \mathbf{b} = 20$ and $\|\mathbf{b}\|^2 = 9$. So the projection scalar is:

$$\frac{20}{9}$$

Now, the vector component is:

$$\frac{20}{9}\mathbf{b} = \frac{20}{9}\langle 1, 2, 2 \rangle = \left\langle \frac{20}{9}, \frac{40}{9}, \frac{40}{9} \right\rangle$$

Thus, the vector component of **a** in the direction of **b** is:

$$\left\langle \frac{20}{9}, \frac{40}{9}, \frac{40}{9} \right\rangle$$

2 Problem 2

2.1 Part (a): Set of Vectors Perpendicular to $\langle 1, 2, 3 \rangle$

The set of vectors perpendicular to $\mathbf{v} = \langle 1, 2, 3 \rangle$ are all vectors $\mathbf{w} = \langle x, y, z \rangle$ that satisfy the dot product equation:

$$\mathbf{v} \cdot \mathbf{w} = 0$$

This gives the equation:

$$1x + 2y + 3z = 0$$

Thus, the set of vectors perpendicular to $\langle 1, 2, 3 \rangle$ forms a plane in $\mathbb{R}^3$ described by the equation:

$$x + 2y + 3z = 0$$

2.2 Part (b): Equation of the Plane with Normal Vector $\langle 1, 2, 3 \rangle$ Passing Through Point $P(4, 5, 6)$

The general equation of a plane with a normal vector $\mathbf{n} = \langle A, B, C \rangle$ and passing through a point $P(x_0, y_0, z_0)$ is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Substituting the normal vector $\mathbf{n} = \langle 1, 2, 3 \rangle$ and the point $P(4, 5, 6)$, we get:

$$1(x - 4) + 2(y - 5) + 3(z - 6) = 0$$

Simplifying:

$$x - 4 + 2y - 10 + 3z - 18 = 0$$

$$x + 2y + 3z = 32$$

Thus, the equation of the plane is:

$$x + 2y + 3z = 32$$

2.3 Part (c): Are the Vector $\langle -5, 1, 1\rangle$ and the Plane $x + 2y + 3z = 6$ Parallel, Perpendicular, Both, or Neither?

The normal vector to the plane $x + 2y + 3z = 6$ is $\mathbf{n} = \langle 1, 2, 3\rangle$. To check if the vector $\mathbf{v} = \langle -5, 1, 1\rangle$ is parallel or perpendicular to the plane, we consider the dot product:

$$\mathbf{n} \cdot \mathbf{v} = 1(-5) + 2(1) + 3(1) = -5 + 2 + 3 = 0$$

Since the dot product is zero, the vector $\mathbf{v}$ is perpendicular to the normal vector, meaning it is **parallel**[1] to the plane.

[1] See, ChatGPT made a mistake here, it said perpendicular at first.

2.4 Part (d): Distance from Point $Q(2, 3, 5)$ to the Plane $x + 2y + 3z = 32$

The distance from a point $Q(x_1, y_1, z_1)$ to a plane $Ax + By + Cz + D = 0$ is given by the formula:

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

For the plane $x + 2y + 3z = 32$, we rewrite it as $x + 2y + 3z - 32 = 0$, so $A = 1$, $B = 2$, $C = 3$, and $D = -32$. The point Q has coordinates $(2, 3, 5)$. Substituting into the formula:

$$\text{Distance} = \frac{|1(2) + 2(3) + 3(5) - 32|}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{|2 + 6 + 15 - 32|}{\sqrt{1 + 4 + 9}} = \frac{|23 - 32|}{\sqrt{14}} = \frac{9}{\sqrt{14}}$$

Thus, the distance from the point $Q(2, 3, 5)$ to the plane is:

$$\frac{9}{\sqrt{14}} \approx 2.41$$

3 Problem 3

Consider the points $P(1, 2, 4)$, $Q(0, 1, 3)$, and $R(2, 4, 7)$.

3.1 Part (a): Calculate the Cross Product $\overrightarrow{PQ} \times \overrightarrow{PR}$

First, we find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \langle 0 - 1, 1 - 2, 3 - 4 \rangle = \langle -1, -1, -1 \rangle$$
$$\overrightarrow{PR} = \langle 2 - 1, 4 - 2, 7 - 4 \rangle = \langle 1, 2, 3 \rangle$$

Now, we compute the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ using the determinant formula:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

Expanding the determinant:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \mathbf{i}((-1)(3) - (-1)(2)) - \mathbf{j}((-1)(3) - (-1)(1)) + \mathbf{k}((-1)(2) - (-1)(1))$$

$$= \mathbf{i}(-3 + 2) - \mathbf{j}(-3 + 1) + \mathbf{k}(-2 + 1)$$

$$= \mathbf{i}(-1) - \mathbf{j}(-2) + \mathbf{k}(-1)$$

$$= \langle -1, 2, -1 \rangle$$

Thus, the cross product is:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, 2, -1 \rangle$$

3.2 Part (b): Equation of the Plane Containing Points P, Q, R

The normal vector to the plane is given by $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, 2, -1 \rangle$. The equation of a plane passing through point $P(1, 2, 4)$ with normal vector $\langle A, B, C \rangle$ is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Substituting $A = -1$, $B = 2$, $C = -1$, and $P(1, 2, 4)$, we get:

$$-1(x - 1) + 2(y - 2) - 1(z - 4) = 0$$

Simplifying:

$$-(x - 1) + 2(y - 2) - (z - 4) = 0$$

$$-x + 1 + 2y - 4 - z + 4 = 0$$

$$-x + 2y - z + 1 = 0$$

Thus, the equation of the plane is:

$$x - 2y + z = 1$$

3.3 Part (c): Area of the Triangle PQR

The area of triangle PQR is given by the formula:

$$\text{Area} = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$$

We already know that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, 2, -1 \rangle$. Now, we compute the magnitude:

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

Thus, the area of the triangle is:

$$\text{Area} = \frac{1}{2}\sqrt{6}$$