Some review problems for Midterm 1

18.02 Recitation MW9

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The power you learned... I can feel it emanating from you. But that's enough talk. Let's get on with why you're here. As the Pokémon League Champion, I accept your challenge!

— *Cynthia in Pokémon Diamond and Pearl*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

§1 Problems

Are you a person that plays every video game on hard mode? Yeah? I have a treat for you.

More seriously, my hope is these problems help you review for Midterm 1 (in less than two weeks!), even if you don't manage to solve them yourself. My suggestion is: think about each for 15-30 minutes, then read the solution (pages 3-end of the online PDF) or come to office hours and I'll explain them. I hope this helps you digest the material; I tried to craft problems that teach deep understanding and piece together multiple ideas, rather than just using one or two isolated recipes.

Solving with friends is encouraged; you'll have more fun thinking together. Still, don't worry even if you solve 0 of the 6 problems; these are way harder than what will actually appear on the midterm. The problems are sorted by the order the topics appeared in class (not by difficulty).

Problem 1. In $\mathbb{R}^3$, compute the projection of the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ onto the plane $x + y + 2z = 0$.

Problem 2. Suppose A, B, C, D are points in $\mathbb{R}^3$. Give a geometric interpretation for this expression:

$$\left|\overrightarrow{DA} \cdot \left(\overrightarrow{DB} \times \overrightarrow{DC}\right)\right|.$$

Problem 3. Fix a plane $\mathcal{P}$ in $\mathbb{R}^3$ which passes through the origin. Consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(\mathbf{v})$ is the projection of $\mathbf{v}$ onto $\mathcal{P}$. Let M denote the 3×3 matrix associated to f. Compute the determinant of M.

Problem 4. Let $\mathbf{a}$ and $\mathbf{b}$ be two perpendicular unit vectors in $\mathbb{R}^3$. A third vector $\mathbf{v}$ in $\mathbb{R}^3$ lies in the span of $\mathbf{a}$ and $\mathbf{b}$. Given that $\mathbf{v} \cdot \mathbf{a} = 2$ and $\mathbf{v} \cdot \mathbf{b} = 3$, compute the magnitudes of the cross products $\mathbf{v} \times \mathbf{a}$ and $\mathbf{v} \times \mathbf{b}$.

Problem 5. Compute the trace of the 2×2 matrix M given the two equations

$$M \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \text{ and } M \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

Problem 6. There are three complex numbers z satisfying $z^3 = 5 + 6i$. Suppose we plot these three numbers in the complex plane. Compute the area of the triangle they enclose.

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§2 Solutions

§2.1 Solution to problem 1

Answer: $\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}$.

§2.1.1 First approach using vector projection

In recitation R02 you had to calculate the distance from a vector to a plane. This problem only requires one step on top of that: you need to then translate by the normal vector. See the cartoon below, where $\mathbf{a}$ denotes the answer.

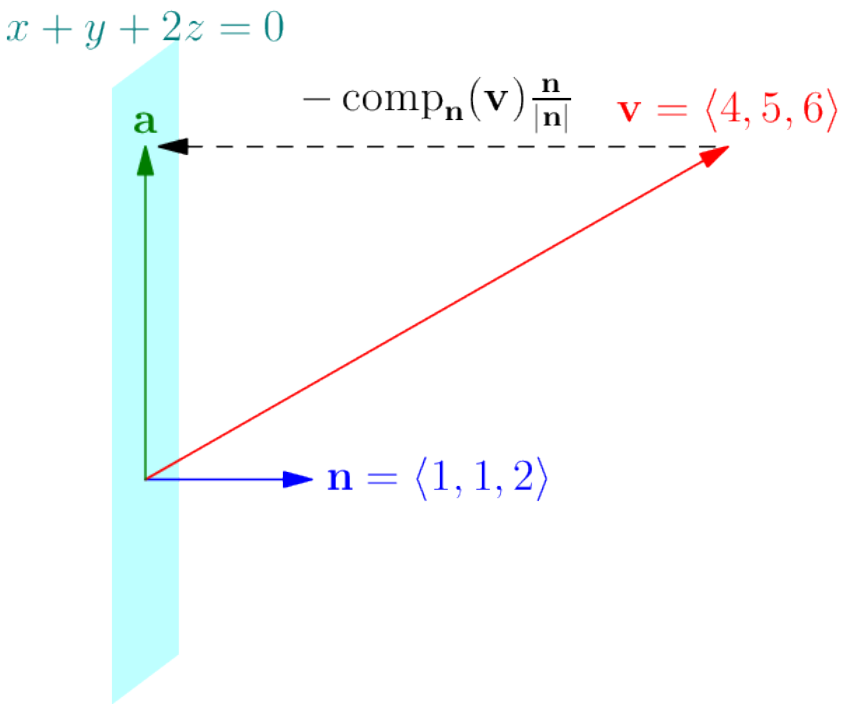

Figure 1: Projection onto a plane.

To execute the calculation, let $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The scalar projection is

$$\operatorname{comp}_{\mathbf{n}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|} = \frac{21}{\sqrt{6}}.$$

The vector projection is then

$$\mathbf{a} = (\operatorname{comp}_{\mathbf{n}}(\mathbf{v}))\frac{\mathbf{n}}{|\mathbf{n}|} = \frac{21}{\sqrt{6}} \frac{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{6}} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 7 \end{pmatrix}.$$

Then the desired projection is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{n}}(\mathbf{v}) = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}.$$

§2.1.2 Second approach using normal vectors only (no projection stuff)

A lot of you don't find vector projection natural (I certainly don't). So it might be easier to imagine shifting $\mathbf{v}$ by *some* multiple of $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and then work out which multiple it is.

Specifically, we're looking for[1] a real number $t \in \mathbb{R}$ such that the vector

$$\mathbf{a} = \mathbf{v} - t\mathbf{n} = \begin{pmatrix} 4-t \\ 5-t \\ 6-2t \end{pmatrix}$$

lies on the plane $x + y + 2z = 0$. But we can actually solve for t just by plugging this $\mathbf{a}$ into the equation of the plane:

$$(4-t) + (5-t) + 2(6-2t) = 0 \Longrightarrow 21 - 6t = 0 \Longrightarrow t = \frac{7}{2}.$$

Hence the answer

$$\mathbf{a} = \begin{pmatrix} 4 - \frac{7}{2} \\ 5 - \frac{7}{2} \\ 6 - 2\left(\frac{7}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}.$$

[1]In comparison to the first solution, the value of t is exactly

$$t = \frac{\mathrm{comp}_{\mathbf{n}}(\mathbf{v})}{|\mathbf{n}|}.$$

But the idea behind the second solution is that you don't *need to know* what the geometric formula of t is. You can just solve for t indirectly by asserting that $\mathbf{a}$ lies on $x + y + 2z = 0$.

§2.2 Solution to problem 2

Answer: This equals the volume of the parallelepiped formed by $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$.

Here are two approaches for proving it.

§2.2.1 First approach using coordinates

Let $D = (0, 0, 0)$, $A = (x_A, y_A, b_A)$, $B = (x_B, y_B, z_B)$, $C = (x_C, y_C, z_C)$. Then expanding the cross product gives

$$(x_A\mathbf{e}_1 + y_A\mathbf{e}_2 + z_A\mathbf{e_3}) \cdot \det\left(\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{pmatrix}\right).$$

If you think about what evaluating the determinant using the formula together with the dot product would give, you should find it's actually just

$$\det\left(\begin{pmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{pmatrix}\right)$$

which is the volume of the parallelepiped.

§2.2.2 Second approach using geometric picture

The cross product $\overrightarrow{DB} \times \overrightarrow{DC}$ is a vector whose area is equal to the parallelogram formed by $\overrightarrow{DB}$ and $\overrightarrow{DC}$. The dot product of that cross product against $\overrightarrow{DA}$ is equal to the *height* of A to plane BCD times this area, and the volume is the height times the area. See the following picture from https://en.wikipedia.org/wiki/Triple_product (in the Wikipedia figure, $\mathbf{a}$ denotes our $\overrightarrow{DA}$, etc.).

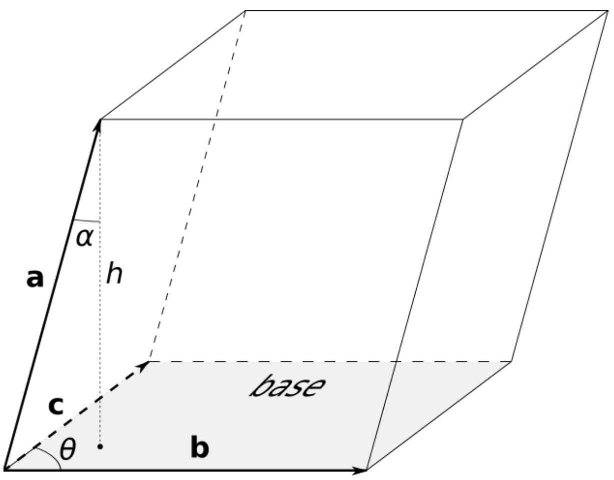

Figure 2: Triple product image taken from Wikipedia.

§2.3 Solution to problem 3

Answer: 0, no matter which plane $\mathcal{P}$ is picked.

§2.3.1 First approach using basis vectors

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the three basis vectors. Then:

- The matrix M is formed by gluing $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, $f(\mathbf{e}_3)$ together.
- I claim the vectors $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, $f(\mathbf{e}_3)$ are linearly dependent. After all, they are all contained in the two-dimensional plane $\mathcal{P}$ by definition, and so three vectors in a plane can't be linearly independent.
- So the determinant is equal to zero (this theorem is one of the criteria we use to check whether vectors are linearly independent or not).

§2.3.2 Second approach using eigenvectors

Let $\mathbf{n}$ be any nonzero normal vector to $\mathcal{P}$. Then $f(\mathbf{n}) = \mathbf{0}$, so $\mathbf{n}$ is an eigenvector with eigenvalue 0. Since the determinant is the product of the eigenvalues, the determinant must be 0 too.

§2.3.3 Third approach using coordinate change

This approach requires you to know the fact that the determinant doesn't change if you rewrite the matrices in a new basis.

Let $\mathbf{n}$ be any nonzero normal vector to $\mathcal{P}$. Pick two more unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ perpendicular to $\mathbf{n}$ that span $\mathcal{P}$. Then $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{n}$ are linearly independent and spanning, i.e. a basis of $\mathbb{R}^3$. So we can change coordinates to use these instead.

We know that

$$\begin{aligned} M(\mathbf{b}_1) &= \mathbf{b}_1 \\ M(\mathbf{b}_2) &= \mathbf{b}_2 \\ M(\mathbf{n}_2) &= \mathbf{0}. \end{aligned}$$

If we wrote M as a matrix *in this new basis* $\langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{n} \rangle$ (rather than the usual basis), we would get the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has determinant 0.

> ***i*** **Remark**
>
> In fact, if you also know that the trace doesn't change when you rewrite M in a different basis, this approach shows the trace M is always exactly $1 + 1 + 0 = 2$ as well, no matter which plane $\mathcal{P}$ is picked.

§2.4 Solution to problem 4

Answer: $|\mathbf{a} \times \mathbf{v}| = 3$ and $|\mathbf{b} \times \mathbf{v}| = 2$.

Since $\mathbf{v}$ is contained in the span of $\mathbf{a}$ and $\mathbf{b}$, we can just pay attention to the plane spanned by these two perpendicular unit vectors. So the geometric picture is that $\mathbf{v}$ can be drawn in a rectangle with $\mathbf{a}$ and $\mathbf{b}$ as a basis, as shown. Because $\mathbf{v} \cdot \mathbf{a} = 2$ and $\mathbf{v} \cdot \mathbf{b} = 3$, this rectangle is 2 by 3.

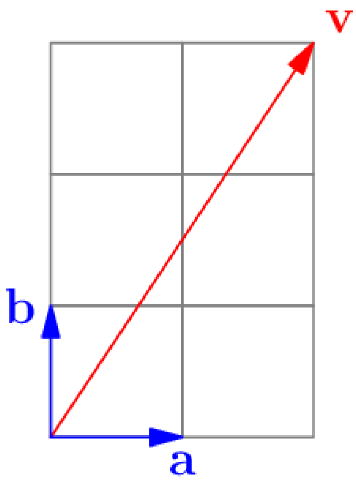

Figure 3: Plotting $\mathbf{v}$ in the span of $\mathbf{a}$ and $\mathbf{b}$.

Now the magnitude of the cross product $\mathbf{a} \times \mathbf{v}$ is supposed to be equal to the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{v}$, which is 3 (because this parallelogram has base $|\mathbf{a}| = 1$ and height $|\mathbf{v} \cdot \mathbf{b}| = 3$). Similarly, $\mathbf{b} \times \mathbf{v}$ has magnitude 2.

§2.5 Solution to problem 5

Answer: 0.

There are several approaches possible. The first two show how to find the four entries of the matrix M. The latter sidestep this entirely and show that the matrix is actually always trace 0.

§2.5.1 First approach: brute force

Like in the pop quiz in my R04 notes, we will try to work out $M\binom{1}{0}$ and $M\binom{0}{1}$. We're looking for constants c_1 and c_2 such that $c_1\binom{4}{7} + c_2\binom{5}{9} = \binom{1}{0}$.

- Solving the system of equations $4c_1 + 5c_2 = 1$ and $7c_1 + 9c_2 = 0$ using your favorite method gives coefficients $c_1 = 9$ and $c_2 = -7$, i.e.

$$9\begin{pmatrix}4\\7\end{pmatrix} - 7\begin{pmatrix}5\\9\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

  This lets us get

$$M\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = 9M\left(\begin{pmatrix}4\\7\end{pmatrix}\right) - 7M\left(\begin{pmatrix}5\\9\end{pmatrix}\right) = 9\begin{pmatrix}5\\9\end{pmatrix} - 7\begin{pmatrix}4\\7\end{pmatrix} = \begin{pmatrix}17\\32\end{pmatrix}.$$

- By solving the analogous system we can find the identity

$$-5\begin{pmatrix}4\\7\end{pmatrix} + 4\begin{pmatrix}5\\9\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix},$$

  and hence:

$$M\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = -5M\left(\begin{pmatrix}4\\7\end{pmatrix}\right) + 4M\left(\begin{pmatrix}5\\9\end{pmatrix}\right) = -5\begin{pmatrix}5\\9\end{pmatrix} + 4\begin{pmatrix}4\\7\end{pmatrix} = \begin{pmatrix}-9\\-17\end{pmatrix}.$$

Gluing these together

$$M = \begin{pmatrix}17 & -9\\32 & -17\end{pmatrix}.$$

The trace is thus $17 + (-17) = 0$.

§2.5.2 Second approach: inverse matrices

We can collate the two given equations into saying that

$$M\begin{pmatrix}4 & 5\\7 & 9\end{pmatrix} = \begin{pmatrix}5 & 4\\9 & 7\end{pmatrix}.$$

Hence one could also recover M by multiplying by the inverse matrix:

$$M = \begin{pmatrix}5 & 4\\9 & 7\end{pmatrix}\begin{pmatrix}4 & 5\\7 & 9\end{pmatrix}^{-1} = \begin{pmatrix}5 & 4\\9 & 7\end{pmatrix}\frac{1}{4\cdot 9 - 7\cdot 5}\begin{pmatrix}9 & -5\\-7 & 4\end{pmatrix} = \begin{pmatrix}17 & -9\\32 & -17\end{pmatrix}.$$

(Of course, we get the same entries for M as the last approach.) Again the trace is $17 + (-17) = 0$.

§2.5.3 Third approach: Guessing eigenvectors and eigenvalues

Let $\mathbf{b}_1 = \binom{4}{7}$ and $\mathbf{b}_2 = \binom{5}{9}$. Adding and subtracting the given equations gives

$$\begin{aligned} M(\mathbf{b}_1 + \mathbf{b}_2) &= \mathbf{b}_1 + \mathbf{b}_2 \\ M(\mathbf{b}_1 - \mathbf{b}_2) &= -(\mathbf{b}_1 - \mathbf{b}_2). \end{aligned}$$

So $\mathbf{b}_1 \pm \mathbf{b}_2$ are eigenvectors with eigenvalues ± 1. Since M is a 2×2 matrix there are at most two eigenvalues: we found them all!

The trace of M is the sum of the eigenvalues. Call in the answer $1 + (-1) = 0$.

§2.5.4 Fourth approach: Change coordinates

This approach requires you to know the fact that the trace doesn't change if you rewrite the matrices in a new basis.

Since $\mathbf{b}_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$ are a basis of $\mathbb{R}^2$, we can change coordinates to use the $\mathbf{b}_i$. In that case,

$$M(\mathbf{b}_1) = \mathbf{b}_2 \quad \text{and} \quad M(\mathbf{b}_2) = \mathbf{b}_1.$$

If we wrote M as a matrix *in this new basis* $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ (rather than the usual basis), we would get the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has trace $0 + 0 = 0$.

§2.6 Solution to problem 6

Answer: $\frac{3\sqrt{3}}{4}\sqrt[3]{61}$.

We start by converting the complex number $5+6i$ into polar form. The modulus r of $5+6i$ is:

$$r = |5+6i| = \sqrt{5^2+6^2} = \sqrt{25+36} = \sqrt{61}.$$

The argument θ is some random angle we won't use the exact value of: $\theta = \arg(5+6i) = \tan^{-1}\left(\frac{6}{5}\right)$.

Now to find the cube roots of $z^3 = 5+6i$, we use the polar form:

$$z = \sqrt[6]{61}\left(\cos\left(\frac{\theta+2k\pi}{3}\right) + i\sin\left(\frac{\theta+2k\pi}{3}\right)\right)$$

for $k = 0, 1, 2$. This gives us three roots corresponding to the different values of k.

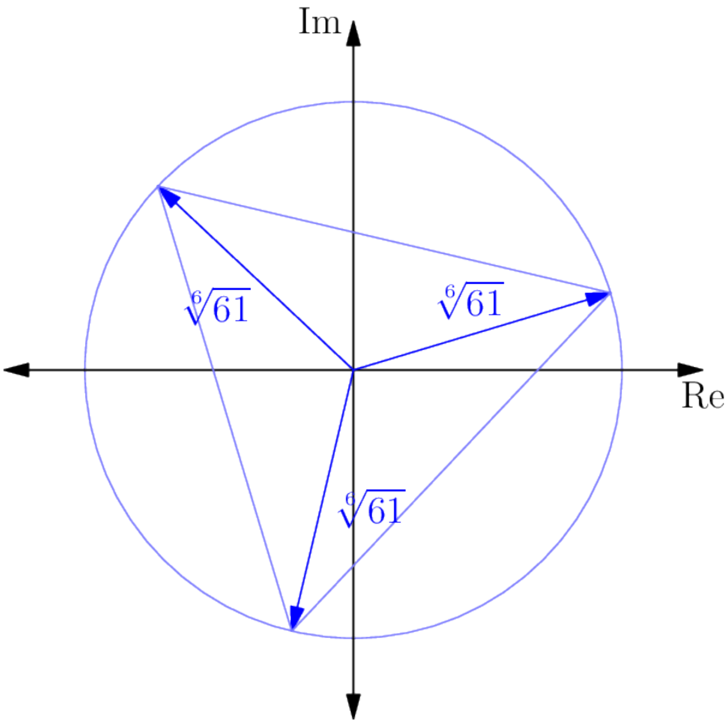

Figure 4: Three solutions to $z^3 = 5+6i$

This looks like an equilateral triangle centered around the origin, where each spoke coming from the origin has magnitude s, where

$$s = \sqrt[6]{61}.$$

If we cut up the equilateral triangle by the three arrows above, we get three small isosceles triangles with a $120°$ angle at the apex. The area of each triangle is going to be $\frac{s^2}{2}\sin(120°)$.

So this gives a final answer of

$$3 \cdot \frac{\sqrt[3]{61}}{2} \cdot \sin(120°) = \frac{3\sqrt{3}}{4}\sqrt[3]{61}.$$