# Some review problems for Midterm 1

## 18.02 Recitation MW9

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*The power you learned... I can feel it emanating from you. But that's enough talk. Let's get on with why you're here. As the Pokémon League Champion, I accept your challenge!*

— *Cynthia in Pokémon Diamond and Pearl*

This handout (and any other DLC's I write) are posted at https://web.evanchen.cc/1802.html.

## §1 Problems

Are you a person that plays every video game on hard mode? Yeah? I have a treat for you.

More seriously, my hope is these problems help you review for Midterm 1 (in less than two weeks!), even if you don't manage to solve them yourself. My suggestion is: think about each for 15-30 minutes, then read the solution (pages 3-end of the online PDF) or come to office hours and I'll explain them. I hope this helps you digest the material; I tried to craft problems that teach deep understanding and piece together multiple ideas, rather than just using one or two isolated recipes.

Solving with friends is encouraged; you'll have more fun thinking together. Still, don't worry even if you solve 0 of the 6 problems; these are way harder than what will actually appear on the midterm. The problems are sorted by the order the topics appeared in class (not by difficulty).

**Problem 1.** In $\mathbb{R}^3$, compute the projection of the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ onto the plane $x + y + 2z = 0$.

**Problem 2.** Suppose $A$, $B$, $C$, $D$ are points in $\mathbb{R}^3$. Give a geometric interpretation for this expression:

$$|\overrightarrow{DA} \cdot (\overrightarrow{DB} \times \overrightarrow{DC})|.$$

**Problem 3.** Fix a plane $\mathcal{P}$ in $\mathbb{R}^3$ which passes through the origin. Consider the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ where $f(\mathbf{v})$ is the projection of $\mathbf{v}$ onto $\mathcal{P}$. Let $M$ denote the $3 \times 3$ matrix associated to $f$. Compute the determinant of $M$.

**Problem 4.** Let $\mathbf{a}$ and $\mathbf{b}$ be two perpendicular unit vectors in $\mathbb{R}^3$. A third vector $\mathbf{v}$ in $\mathbb{R}^3$ lies in the span of $\mathbf{a}$ and $\mathbf{b}$. Given that $\mathbf{v} \cdot \mathbf{a} = 2$ and $\mathbf{v} \cdot \mathbf{b} = 3$, compute the magnitudes of the cross products $\mathbf{v} \times \mathbf{a}$ and $\mathbf{v} \times \mathbf{b}$.

**Problem 5.** Compute the trace of the $2 \times 2$ matrix $M$ given the two equations

$$M \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \text{ and } M \begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

**Problem 6.** There are three complex numbers $z$ satisfying $z^3 = 5 + 6i$. Suppose we plot these three numbers in the complex plane. Compute the area of the triangle they enclose.

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# §2 Solutions

## §2.1 Solution to problem 1

**Answer**: $\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}$.

### §2.1.1 First approach using vector projection

In recitation R02 you had to calculate the distance from a vector to a plane. This problem only requires one step on top of that: you need to then translate by the normal vector. See the cartoon below, where $\mathbf{a}$ denotes the answer.

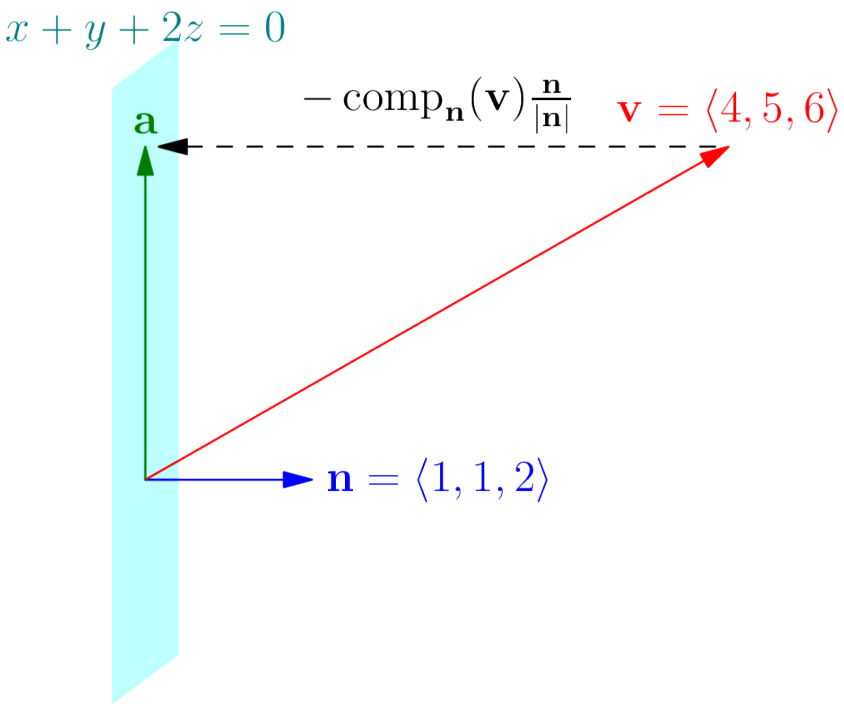


Figure 1: Projection onto a plane.

To execute the calculation, let $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The scalar projection is

$$\operatorname{comp}_{\mathbf{n}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|} = \frac{21}{\sqrt{6}}.$$

The vector projection is then

$$\mathbf{a} = (\operatorname{comp}_{\mathbf{n}}(\mathbf{v}))\frac{\mathbf{n}}{|\mathbf{n}|} = \frac{21}{\sqrt{6}} \frac{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{6}} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 7 \end{pmatrix}.$$

Then the desired projection is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{n}}(\mathbf{v}) = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}.$$

### §2.1.2 Second approach using normal vectors only (no projection stuff)

A lot of you don't find vector projection natural (I certainly don't). So it might be easier to imagine shifting $\mathbf{v}$ by *some* multiple of $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and then work out which multiple it is.

Specifically, we're looking for[1] a real number $t \in \mathbb{R}$ such that the vector

$$\mathbf{a} = \mathbf{v} - t\mathbf{n} = \begin{pmatrix} 4-t \\ 5-t \\ 6-2t \end{pmatrix}$$

lies on the plane $x + y + 2z = 0$. But we can actually solve for $t$ just by plugging this $\mathbf{a}$ into the equation of the plane:

$$(4-t) + (5-t) + 2(6-2t) = 0 \Longrightarrow 21 - 6t = 0 \Longrightarrow t = \frac{7}{2}.$$

Hence the answer

$$\mathbf{a} = \begin{pmatrix} 4 - \frac{7}{2} \\ 5 - \frac{7}{2} \\ 6 - 2\left(\frac{7}{2}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{pmatrix}.$$

[1] In comparison to the first solution, the value of $t$ is exactly

$$t = \frac{\operatorname{comp}_{\mathbf{n}}(\mathbf{v})}{|\mathbf{n}|}.$$

But the idea behind the second solution is that you don't *need to know* what the geometric formula of $t$ is. You can just solve for $t$ indirectly by asserting that $\mathbf{a}$ lies on $x + y + 2z = 0$.

## §2.2 Solution to problem 2

**Answer**: This equals the volume of the parallelepiped formed by $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$.

Here are two approaches for proving it.

### §2.2.1 First approach using coordinates

Let $D = (0,0,0)$, $A = (x_A, y_A, b_A)$, $B = (x_B, y_B, z_B)$, $C = (x_C, y_C, z_C)$. Then expanding the cross product gives

$$(x_A \mathbf{e}_1 + y_A \mathbf{e}_2 + z_A \mathbf{e_3}) \cdot \det \left( \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{pmatrix} \right).$$

If you think about what evaluating the determinant using the formula together with the dot product would give, you should find it's actually just

$$\det \left( \begin{pmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{pmatrix} \right)$$

which is the volume of the parallelepiped.

### §2.2.2 Second approach using geometric picture

The cross product $\overrightarrow{DB} \times \overrightarrow{DC}$ is a vector whose area is equal to the parallelogram formed by $\overrightarrow{DB}$ and $\overrightarrow{DC}$. The dot product of that cross product against $\overrightarrow{DA}$ is equal to the *height* of $A$ to plane $BCD$ times this area, and the volume is the height times the area. See the following picture from https://en.wikipedia.org/wiki/Triple_product (in the Wikipedia figure, $\mathbf{a}$ denotes our $\overrightarrow{DA}$, etc.).

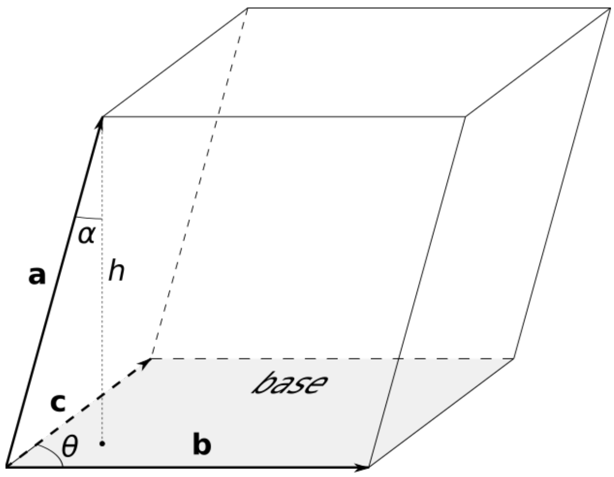


Figure 2: Triple product image taken from Wikipedia.

## §2.3 Solution to problem 3

**Answer**: 0, no matter which plane $\mathcal{P}$ is picked.

### §2.3.1 First approach using basis vectors

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the three basis vectors. Then:

- The matrix $M$ is formed by gluing $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, $f(\mathbf{e}_3)$ together.
- I claim the vectors $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, $f(\mathbf{e}_3)$ are linearly dependent. After all, they are all contained in the two-dimensional plane $\mathcal{P}$ by definition, and so three vectors in a plane can't be linearly independent.
- So the determinant is equal to zero (this theorem is one of the criteria we use to check whether vectors are linearly independent or not).

### §2.3.2 Second approach using eigenvectors

Let $\mathbf{n}$ be any nonzero normal vector to $\mathcal{P}$. Then $f(\mathbf{n}) = \mathbf{0}$, so $\mathbf{n}$ is an eigenvector with eigenvalue 0. Since the determinant is the product of the eigenvalues, the determinant must be 0 too.

### §2.3.3 Third approach using coordinate change

This approach requires you to know the fact that the determinant doesn't change if you rewrite the matrices in a new basis.

Let $\mathbf{n}$ be any nonzero normal vector to $\mathcal{P}$. Pick two more unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ perpendicular to $\mathbf{n}$ that span $\mathcal{P}$. Then $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{n}$ are linearly independent and spanning, i.e. a basis of $\mathbb{R}^3$. So we can change coordinates to use these instead.

We know that

$$\begin{aligned} M(\mathbf{b}_1) &= \mathbf{b}_1 \\ M(\mathbf{b}_2) &= \mathbf{b}_2 \\ M(\mathbf{n}_2) &= \mathbf{0}. \end{aligned}$$

If we wrote $M$ as a matrix *in this new basis* $\langle \mathbf{b}_1, \mathbf{b}_2, \mathbf{n} \rangle$ (rather than the usual basis), we would get the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has determinant 0.

**Remark**

In fact, if you also know that the trace doesn't change when you rewrite $M$ in a different basis, this approach shows the trace $M$ is always exactly $1 + 1 + 0 = 2$ as well, no matter which plane $\mathcal{P}$ is picked.

## §2.4 Solution to problem 4

**Answer**: $|\mathbf{a} \times \mathbf{v}| = 3$ and $|\mathbf{b} \times \mathbf{v}| = 2$.

Since $\mathbf{v}$ is contained in the span of $\mathbf{a}$ and $\mathbf{b}$, we can just pay attention to the plane spanned by these two perpendicular unit vectors. So the geometric picture is that $\mathbf{v}$ can be drawn in a rectangle with $\mathbf{a}$ and $\mathbf{b}$ as a basis, as shown. Because $\mathbf{v} \cdot \mathbf{a} = 2$ and $\mathbf{v} \cdot \mathbf{b} = 3$, this rectangle is 2 by 3.

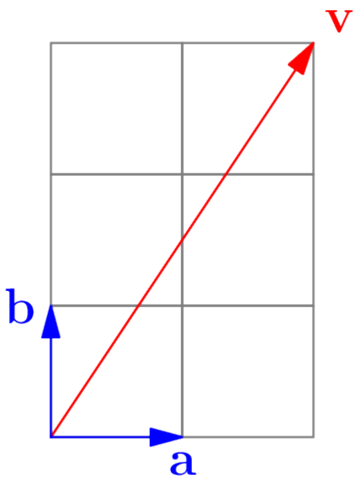


Figure 3: Plotting $\mathbf{v}$ in the span of $\mathbf{a}$ and $\mathbf{b}$.

Now the magnitude of the cross product $\mathbf{a} \times \mathbf{v}$ is supposed to be equal to the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{v}$, which is 3 (because this parallelogram has base $|\mathbf{a}| = 1$ and height $|\mathbf{v} \cdot \mathbf{b}| = 3$). Similarly, $\mathbf{b} \times \mathbf{v}$ has magnitude 2.

## §2.5 Solution to problem 5

**Answer**: 0.

There are several approaches possible. The first two show how to find the four entries of the matrix $M$. The latter sidestep this entirely and show that the matrix is actually always trace 0.

### §2.5.1 First approach: brute force

Like in the pop quiz in my R04 notes, we will try to work out $M\binom{1}{0}$ and $M\binom{0}{1}$. We're looking for constants $c_1$ and $c_2$ such that $c_1\binom{4}{7} + c_2\binom{5}{9} = \binom{1}{0}$.

- Solving the system of equations $4c_1 + 5c_2 = 1$ and $7c_1 + 9c_2 = 0$ using your favorite method gives coefficients $c_1 = 9$ and $c_2 = -7$, i.e.

$$9\begin{pmatrix} 4 \\ 7 \end{pmatrix} - 7\begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  This lets us get

$$M\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 9M\left(\begin{pmatrix} 4 \\ 7 \end{pmatrix}\right) - 7M\left(\begin{pmatrix} 5 \\ 9 \end{pmatrix}\right) = 9\begin{pmatrix} 5 \\ 9 \end{pmatrix} - 7\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 17 \\ 32 \end{pmatrix}.$$

- By solving the analogous system we can find the identity

$$-5\begin{pmatrix} 4 \\ 7 \end{pmatrix} + 4\begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

  and hence:

$$M\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = -5M\left(\begin{pmatrix} 4 \\ 7 \end{pmatrix}\right) + 4M\left(\begin{pmatrix} 5 \\ 9 \end{pmatrix}\right) = -5\begin{pmatrix} 5 \\ 9 \end{pmatrix} + 4\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -9 \\ -17 \end{pmatrix}.$$

Gluing these together

$$M = \begin{pmatrix} 17 & -9 \\ 32 & -17 \end{pmatrix}.$$

The trace is thus $17 + (-17) = 0$.

### §2.5.2 Second approach: inverse matrices

We can collate the two given equations into saying that

$$M\begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 9 & 7 \end{pmatrix}.$$

Hence one could also recover $M$ by multiplying by the inverse matrix:

$$M = \begin{pmatrix} 5 & 4 \\ 9 & 7 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 4 \\ 9 & 7 \end{pmatrix}\frac{1}{4 \cdot 9 - 7 \cdot 5}\begin{pmatrix} 9 & -5 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 17 & -9 \\ 32 & -17 \end{pmatrix}.$$

(Of course, we get the same entries for $M$ as the last approach.) Again the trace is $17 + (-17) = 0$.

### §2.5.3 Third approach: Guessing eigenvectors and eigenvalues

Let $\mathbf{b}_1 = \binom{4}{7}$ and $\mathbf{b}_2 = \binom{5}{9}$. Adding and subtracting the given equations gives

$$\begin{aligned} M(\mathbf{b}_1 + \mathbf{b}_2) &= \mathbf{b}_1 + \mathbf{b}_2 \\ M(\mathbf{b}_1 - \mathbf{b}_2) &= -(\mathbf{b}_1 - \mathbf{b}_2). \end{aligned}$$

So $\mathbf{b}_1 \pm \mathbf{b}_2$ are eigenvectors with eigenvalues $\pm 1$. Since $M$ is a $2 \times 2$ matrix there are at most two eigenvalues: we found them all!

The trace of $M$ is the sum of the eigenvalues. Call in the answer $1 + (-1) = 0$.

### §2.5.4 Fourth approach: Change coordinates

This approach requires you to know the fact that the trace doesn't change if you rewrite the matrices in a new basis.

Since $\mathbf{b}_1 = \binom{4}{7}$ and $\mathbf{b}_2 = \binom{5}{9}$ are a basis of $\mathbb{R}^2$, we can change coordinates to use the $\mathbf{b}_i$. In that case,

$$M(\mathbf{b}_1) = \mathbf{b}_2 \quad \text{and} \quad M(\mathbf{b}_2) = \mathbf{b}_1.$$

If we wrote $M$ as a matrix *in this new basis* $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle$ (rather than the usual basis), we would get the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has trace $0 + 0 = 0$.

## §2.6 Solution to problem 6

**Answer**: $\frac{3\sqrt{3}}{4}\sqrt[3]{61}$.

We start by converting the complex number $5+6i$ into polar form. The modulus $r$ of $5+6i$ is:

$$r = |5+6i| = \sqrt{5^2+6^2} = \sqrt{25+36} = \sqrt{61}.$$

The argument $\theta$ is some random angle we won't use the exact value of: $\theta = \arg(5+6i) = \tan^{-1}\left(\frac{6}{5}\right)$.

Now to find the cube roots of $z^3 = 5+6i$, we use the polar form:

$$z = \sqrt[6]{61}\left(\cos\left(\frac{\theta+2k\pi}{3}\right) + i\sin\left(\frac{\theta+2k\pi}{3}\right)\right)$$

for $k = 0, 1, 2$. This gives us three roots corresponding to the different values of $k$.

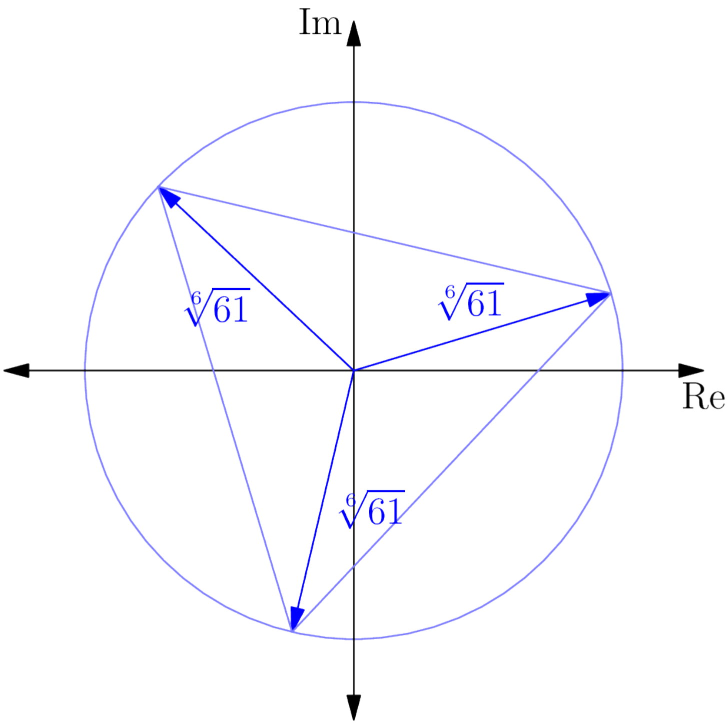


Figure 4: Three solutions to $z^3 = 5+6i$

This looks like an equilateral triangle centered around the origin, where each spoke coming from the origin has magnitude $s$, where

$$s = \sqrt[6]{61}.$$

If we cut up the equilateral triangle by the three arrows above, we get three small isosceles triangles with a $120°$ angle at the apex. The area of each triangle is going to be $\frac{s^2}{2}\sin(120°)$.

So this gives a final answer of

$$3 \cdot \frac{\sqrt[3]{61}}{2} \cdot \sin(120°) = \frac{3\sqrt{3}}{4}\sqrt[3]{61}.$$