JMO 2026/3

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Twitch Solves ISL

Episode 178

Problem

Let ABC be an acute scalene triangle with no angle equal to 60°. Let ω be the circumcircle of ABC. Let Δ_B be the equilateral triangle with three vertices on ω, one of which is B. Let ℓ_B be the line through the two vertices of Δ_B other than B. Let Δ_C and ℓ_C be defined analogously. Let Y be the intersection of AC and ℓ_B, and let Z be the intersection of AB and ℓ_C.

Suppose that the circumcircle of AYZ intersects ω at $P \neq A$, BC intersects YZ at D, and PA intersects YZ at E. Prove that $PE = PD$.

Video

https://youtu.be/fGN283EHQaU

External Link

https://aops.com/community/p37578105

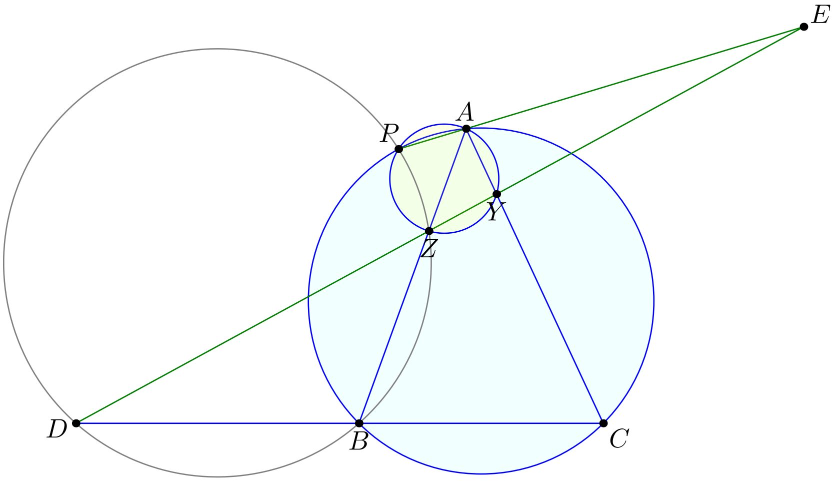
Solution

Because P is the Miquel point of $BZYC$, it follows $DBZP$ is cyclic. Hence

$$\begin{aligned} \measuredangle PDE &= \measuredangle PDZ = \measuredangle PBZ = \measuredangle PBA \\ \measuredangle DEP &= \measuredangle(\overline{YZ}, \overline{AP}). \end{aligned}$$

So the problem is solved if we can show

$$\measuredangle PBA = \measuredangle(\overline{YZ}, \overline{AP}). \qquad (\bigstar)$$



We use complex numbers to prove $(\bigstar)$. Let $\omega = e^{2\pi i/3}$. Since Y is the intersection of a, c, ωb and $\omega^2 b$, we have

$$\begin{aligned} y &= \frac{b^2(a+c) - ac(\omega b + \omega^2 b)}{b^2 - ac} = \frac{b \cdot (ab+bc+ca)}{b^2 - ac}. \\ z &= \frac{c \cdot (ab+bc+ca)}{c^2 - ab}. \end{aligned}$$

We compute the point P now.

Claim. We have

$$p = \frac{ab+bc+ca}{a+b+c}.$$

Proof. Note that

$$\frac{p-z}{p-y} = \frac{p-b}{p-c} \iff p = \frac{by - cz}{b+y-c-z}.$$

The numerator and denominator are respectively

$$\begin{aligned} by - cz &= (ab+bc+ca) \cdot \left(\frac{b^2}{b^2-ac} - \frac{c^2}{c^2-ab} \right) \\ &= (ab+bc+ca) \cdot \frac{a(c^3-b^3)}{(b^2-ac)(c^2-ab)} \\ &= -(ab+bc+ca) \cdot \frac{a(b-c)(b^2+bc+c^2)}{(b^2-ac)(c^2-ab)} \\ b+y-c-z &= \frac{(b-c)[(b^2-ac)(c^2-ab) - (ab+bc+ca)^2]}{(b^2-ac)(c^2-ab)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(b-c)[-ab^3-ac^3+a^2bc-a^2b^2-c^2a^2-2abc(a+b+c)]}{(b^2-ac)(c^2-ab)} \\
&= \frac{-a(b-c)[a(b^2+bc+c^2)+b^3+c^3+2bc(b+c)]}{(b^2-ac)(c^2-ab)} \\
&= \frac{-a(b-c)(a+b+c)(b^2+bc+c^2)}{(b^2-ac)(c^2-ab)}.
\end{aligned}$$

Dividing gives the conclusion. □

The desired conclusion (★) is encoded as

$$\mathbb{R} \ni \frac{p}{p+a} \div \frac{y-z}{p-a} = \frac{p-a}{p+a} \cdot \frac{p}{y-z} \iff \frac{p}{y-z} \in i\mathbb{R}$$

since $\frac{p-a}{p+a}$ is obviously pure imaginary (for $|a| = |p| = 1$). We compute

$$\begin{aligned}
y-z &= (ab+bc+ca) \cdot \frac{b(c^2-ab)-c(b^2-ac)}{(b^2-ac)(c^2-ab)} \\
&= (ab+bc+ca) \cdot \frac{-(b-c)(bc+ab+ac)}{(b^2-ac)(c^2-ab)} \\
&= -(ab+bc+ca)^2 \cdot \frac{b-c}{(b^2-ac)(c^2-ab)}
\end{aligned}$$

Hence

$$\frac{p}{y-z} = \frac{(b^2-ac)(c^2-ab)}{(b-c)(a+b+c)(ab+bc+ca)}.$$

The conjugate is

$$\overline{\left(\frac{p}{y-z}\right)} = \frac{\left(\frac{1}{b^2}-\frac{1}{ac}\right)\left(\frac{1}{c^2}-\frac{1}{ab}\right)}{\left(\frac{1}{b}-\frac{1}{c}\right) \cdot \frac{ab+bc+ca}{abc} \cdot \frac{a+b+c}{abc}} = -\frac{p}{y-z}$$

as desired.