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TWITCH SOLVES ISL

Episode 178

Problem

Let p be a prime number and let n be a positive integer such that p does not divide n. Denote by k the number of positive divisors of n, and by $1 = d_1 < d_2 < \cdots < d_k = n$ the positive divisors of n. For $i = 1, 2, \dots, k$, let c_i be the number of positive divisors ℓ of d_i^2 such that $d_i - \ell$ is divisible by p. Prove that

$$(p-1)(c_1 + c_2 + \cdots + c_k) \geq k^2.$$

Video

https://youtu.be/BJaMuBJcNJA

External Link

https://aops.com/community/p37802621

Solution

Set $n = q_1^{e_1} \dots q_m^{e_m}$, where each q_i is prime. Fix a primitive root g modulo p, and let $q_i \equiv g^{\lambda_i} \pmod p$. In what follows, index variables i will always range from 1 through m (in contrast to the problem statement).

We first rephrase how to think about c_i:

Claim (Rephrasing of c_j)**.** Suppose d_j is the divisor $q_1^{y_1} \dots q_m^{y_m}$. Then c_j counts the number of m-tuples $(x_1, \dots, x_m) \in \mathbb{Z}^m$ such that $|x_i| \le y_i$, and $\sum \lambda_i x_i \equiv 0 \pmod{p-1}$.

Proof. Explicitly, in the original problem notation, $\ell = d_i \cdot \prod q_i^{x_i}$. ☐

Hence, we are motivated to double-count $\sum c_i$ by the x_i's instead.

Claim (Double counting over x_i's)**.** We have

$$\sum c_i = \sum_{\substack{-e_i \le x_i \le e_i \\ \sum \lambda_i x_i \equiv 0 \bmod p-1}} \prod_{i=1}^{m} (e_i + 1 - |x_i|).$$

Proof. This is exactly the double counting described just now; for a particular tuple $(x_1, \dots, x_m)$ the number of choices of y_i is $e_i + 1 - |x_i|$ for each i. ☐

The second main ingredient is to then rewrite using a roots of unity filter. We know that for any integer s, we have

$$\sum_{\omega^{p-1}=1} \omega^s = \begin{cases} 1 & s \equiv 0 \pmod{p-1} \\ 0 & s \not\equiv 0 \pmod{p-1}. \end{cases}$$

Hence, we bring this in:

$$\begin{aligned} \sum c_i &= \sum_{\substack{-e_i \le x_i \le e_i \\ \sum \lambda_i x_i \equiv 0 \bmod p-1}} \prod_{i=1}^{m} (e_i + 1 - |x_i|) \\ &= \sum_{-e_i \le x \le e_i} \frac{\sum_{\omega^{p-1}=1} \omega^{\sum \lambda_i x_i}}{p-1} \prod_{i=1}^{m} (e_i + 1 - |x_i|) \\ &= \frac{1}{p-1} \sum_{\omega^{p-1}=1} \sum_{-e_i \le x_i \le e_i} \prod_{i=1}^{m} \omega^{\lambda_i x_i} (e_i + 1 - |x_i|) \\ &= \frac{1}{p-1} \sum_{\omega^{p-1}=1} \prod_{i=1}^{m} \left(\sum_{-e_i \le x_i \le e_i} \omega^{\lambda_i x_i} (e_i + 1 - |x_i|) \right). \end{aligned}$$

The inner sum might be dense in notation, but in fact it's easy to describe explicitly:

Lemma. If z is a complex number with $|z| = 1$ and $e \ge 1$ is an integer, then

$$\sum_{x=-e}^{e} z^x \cdot (e + 1 - |x|) = |1 + z + \dots + z^e|^2.$$

Proof. If we write out the sum notation, the left-hand side might read more naturally as.

$$z^{-e} + 2z^{-e} + \dots + ez^{-1} + (e+1) + ez + (e-1)z^2 + \dots + z^e.$$

The point is that it factors into

$$(1 + z + \dots + z^e)(1 + z^{-1} + \dots + z^{-e}).$$

For $|z| = 1$ we thus get $1 + z + \dots + z^e$ times its own conjugate. ☐

Applying that we finally arrive at

$$\sum c_i = \frac{1}{p-1} \sum_{\omega^{p-1}=1} \prod_{i=1}^{m} \left| 1 + \omega^{\lambda_i} + \dots + \omega^{\lambda_i e_i} \right|^2 .$$

We consider only the contribution of $\omega = 1$ and simply note the other terms are nonnegative to get the estimate

$$\sum c_i \geq \frac{1}{p-1} (e_1 + 1)^2 \dots (e_m + 1)^2$$

which is what we wanted.