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TWITCH SOLVES ISL

Episode 177

Problem

Let a and b be positive integers such that $\varphi(ab + 1)$ divides $a^2 + b^2 + 1$. Prove that a and b are Fibonacci numbers.

External Link

<https://aops.com/community/p37586266>

Solution

We start with the following.

Claim. The number $ab + 1$ must be a prime power.

Proof. Note to start that:

- Since Euler phi only gives even values, $2 \mid a^2 + b^2 + 1$.
- Hence, exactly one of a and b is odd.
- Thus, $a^2 + b^2 + 1 \equiv 2 \pmod{4}$.

So $\nu_2(\varphi(ab + 1)) = 1$, and hence $ab + 1$ must have at most one prime factor. \square

Henceforth, let $ab + 1 = p^e$. The $e = 1$ case is easy, and does not even rely on $ab + 1$ being prime:

Claim. The equation $ab + 1 \mid a^2 + b^2 + 1$ is equivalent to $\{a, b\} = \{F_{2k-1}, F_{2k+1}\}$ for $k \geq 0$. Here the Fibonacci sequence F_i is indexed by $F_{-1} = F_1 = F_2 = 1$ and $F_0 = 0$.

Proof. Classical Vieta jumping argument. \square

We now consider $e \geq 2$. In that case, we have

$$\begin{aligned} \varphi(p^e) &= p^{e-1} \cdot (p - 1) \mid a^2 + b^2 + 1 \\ p^e - 1 &= ab. \end{aligned}$$

This implies that

$$0 \equiv a^2 + \left(\frac{p^e - 1}{a}\right)^2 + 1 \equiv \frac{a^4 + a^2 + 1}{a^2} = \frac{(a^2 + a + 1)(a^2 - a + 1)}{a^2} \pmod{p^{e-1}}.$$

In particular, this implies $p = 3$ or $p \equiv 1 \pmod{3}$.

- If $p \equiv 1 \pmod{3}$, then we get $3 \mid p - 1 \mid a^2 + b^2 + 1$, which forces a and b to be nonzero modulo 3. However, then $p^e - 1 = ab \not\equiv 0 \pmod{3}$, which is impossible.
- Now suppose $p = 3$. Since $x^2 \pm x + 1$ is never divisible by 9, this forces $e \leq 2$. Hence, we only have to consider the case $ab = 8$. We find there is one additional solution here only: $(1, 8)$, and both 1 and 8 are Fibonacci numbers.