USAMO 2026/1

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Twitch Solves ISL

Episode 177

Problem

Fix an integer $n \geq 2$. For which real numbers x is

$$\lfloor nx \rfloor - \sum_{k=1}^{n} \frac{\lfloor kx \rfloor}{k}$$

maximal, and what is the maximal value that this expression can take?

External Link

https://aops.com/community/p37578108

Solution

Answer. The answer is $\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ and is achieved exactly when the fractional part of x is at least $1 - 1/n$.

Cosmetic rewriting. Since the function is invariant under shifting $x \mapsto x + 1$, we will focus only on $x = 1 - y$ for $0 < y \le 1$. In that case, the sum in the problem can be rewritten as

$$\begin{aligned} S(y) &:= \lfloor n(1-y) \rfloor - \sum_{k=1}^{n} \frac{\lfloor k(1-y) \rfloor}{k} \\ &= n - \lceil ny \rceil - \sum_{k=1}^{n} \left(1 - \frac{\lceil ky \rceil}{k} \right) \\ &= \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \right) - \lceil ny \rceil + \sum_{k=1}^{n} \frac{\lceil ky \rceil - 1}{k} \\ &= \left(\frac{1}{2} + \dots + \frac{1}{n} \right) - \left[(\lceil ny \rceil - 1) - \sum_{k=1}^{n} \frac{\lceil ky \rceil - 1}{k} \right]. \end{aligned}$$

Hence, if we let

$$T(y) = (\lceil ny \rceil - 1) - \sum_{k=1}^{n} \frac{\lceil ky \rceil - 1}{k}$$

be the bracketed term, it suffices to show $T(y) \ge 0$ with equality only when $0 < y \le \frac{1}{n}$.

Main proof. Let $p := \lceil ny \rceil - 1$. If $p = 0$ (hence if $0 < y \le 1/n$), then $T(y) = 0$, as advertised.

Now assume $p > 0$, so that $\frac{p}{n} < y \le \frac{p+1}{n}$. We solve the problem once we prove:

Claim (Essentially USAMO 1981/5)**.** For $1 \le p < n$, we have the inequality

$$\sum_{k=1}^{n} \frac{\left\lceil k \cdot \frac{p+1}{n} \right\rceil - 1}{k} < p.$$

Proof. We rewrite the left-hand side as

$$\begin{aligned} \sum_{k=1}^{n} \frac{\left\lceil k \cdot \frac{p+1}{n} \right\rceil - 1}{k} &= \sum_{k=1}^{n} \frac{\sum_{1 \le d < k \cdot \frac{p+1}{n}} 1}{k} \\ &= \sum_{1 \le d \le p} \left(\sum_{d \cdot \frac{n}{p+1} < k \le n} \frac{1}{k} \right) \\ &= \left(\sum_{\frac{1}{p+1} n < k \le \frac{2}{p+1} n} \frac{1}{k} \right) + 2 \left(\sum_{\frac{2}{p+1} n < k \le \frac{3}{p+1} n} \frac{1}{k} \right) + 3 \left(\sum_{\frac{3}{p+1} n < k \le \frac{4}{p+1} n} \frac{1}{k} \right) \\ &\qquad + \dots + p \left(\sum_{\frac{p}{p+1} n < k \le \frac{p+1}{p+1} n} \frac{1}{k} \right). \end{aligned}$$

Each of the sums above has at most $\frac{n}{p+1}$ terms in it. We bound each of the fractions by the lower bound for k, to get a strict upper bound of

$$\frac{n}{p+1}\left(\frac{1}{\frac{1}{p+1}n}+2\cdot\frac{1}{\frac{2}{p+1}n}+\cdots+p\cdot\frac{1}{\frac{p}{p+1}n}\right)=p. \qquad \square$$