

# USAMO 2026/4

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TWITCH SOLVES ISL

Episode 176

## Problem

A positive integer  $n$  is called *solitary* if, for any non-negative integers  $a$  and  $b$  such that  $a + b = n$ , either  $a$  or  $b$  contains the digit “1”. Determine, with proof, the number of solitary integers less than  $10^{2026}$ .

## External Link

<https://aops.com/community/p37586238>

## Solution

We claim that a number is solitary if and only if all the following hold:

- The digit 1 appears exactly once;
- Every digit (possibly none) to the left of the 1 is 0 or 2;
- Every digit (possibly none) to the right of the 1 is 9.

For example, 202201999999 is solitary.

**Proof all such numbers are solitary.** The basic idea is to use induction as follows: If the last digit of  $n$  is 9, then the last digits of  $a$  and  $b$  sum to 9; hence we can ignore the last digit altogether. Thus, we reduce to the case where the last digit is 1.

We continue the induction in a similar way in this situation. Zero-pad all the numbers to be the same length as  $n$ . Take the leading 2 of  $n$ .

- If either  $a$  or  $b$  has a leading digit 2, we can delete it and continue the induction.
- Otherwise, clearly one of  $a$  or  $b$  must have leading digit 1, as needed.

**Proof that every solitary number is of this form.** Let  $n$  be solitary. In all the diagrams that follow, ellipses denote groups of digits other than 1 (possibly none).

We first reduce to the case where  $n$  has exactly one 1:

- By taking  $b = 0$ , we see there is at least a single 1.
- Suppose  $n$  has an even number of 1's. The idea is to pair the 1's using blocks like  $9\dots 9$ , as in the following diagram:

$$\begin{array}{rcccccc} n = & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots \\ a = & 0000 & 9993 & 0000 & 9993 & 0000 \\ b = & \dots 0 & \dots 8 & 0000 & \dots 8 & \dots \end{array}$$

- Next, suppose  $n$  has an odd number of 1's and at least three 1's. The strategy is similar:

$$\begin{array}{rcccccc} n = & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots 1 & \dots \\ a = & 0000 & 9993 & 9993 & 0000 & 9993 & 0000 \\ b = & \dots 1 & \dots 7 & \dots 8 & 0000 & \dots 8 & \dots \end{array}$$

So, assume  $n$  has exactly one 1 appear.

- If a digit  $b \neq 0, 9$  appears to the left of 1, let  $c = b + 1$  to get

$$\begin{array}{rcccc} n = & \dots & 1 & \dots b & \dots \\ a = & 0000 & 0 & 9999 & 0000 \\ b = & \dots & 0 & \dots c & \dots \end{array}$$

If  $b = 0$ , one can instead replace 9999 with 9998 and choose  $c = 2$ .

- Finally, if a digit  $d \neq 0, 2$  appears to the left of 1, let  $e = d - 1$  and use a bunch of 9's:

$$\begin{array}{rcccc} n = & \dots d & \dots & 1 & \dots \\ a = & 0000 & 9999 & 9 & 0000 \\ b = & \dots e & \dots & 2 & \dots \end{array}$$

**Final count.** Zero-pad the number so that it has exactly 2026 digits. If we pick the digit 1 to appear at the  $i$ 'th place from the left, for  $1 \leq i \leq 2026$ , then there are  $2^{i-1}$  ways to pick the first  $i - 1$  digits (either 0 or 2); the rest are all 9. Hence, the answer is

$$\sum_{i=1}^{2026} 2^{i-1} = \boxed{2^{2026} - 1}.$$