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Twitch Solves ISL

Episode 175

Problem

Let $a > 1$ be an integer. Prove there exists an integer $n \geq 1$ such that when $2^n a$ is written in decimal, there are at least as many 8's as 4's in the decimal representation of 2^n.

Solution

Pick a large integer X to be determined later, and imagine writing a, $2a$, $4a$, …, $2^{X-1}a$ in a right-aligned table, with a column for each digit. For example, $a = 1337$ we have

$$\begin{bmatrix} & 1 & 3 & 3 & 7 \\ & 2 & 6 & 7 & 4 \\ & 5 & 3 & 5 & 8 \\ 1 & 0 & 7 & 1 & 6 \\ 2 & 1 & 4 & 3 & 2 \\ 4 & 2 & 8 & 6 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Observe that for large X, the number of columns is $\frac{X}{\log_2(10)} + O_a(1)$.

The main observation about the columns is:

Claim. In any column, if a 4 appears, then it is followed by zero or more 9's and then an 8. In particular, within any column, the number of 4's is at most one more than the number of 8's.

Proof. Check directly. □

Assume for contradiction the result is not true and take X large enough. Let $N(4)$ and $N(8)$ denote the number of 4's and 8's in the whole table.

- On the one hand, counting by rows, the contradiction assumption ensures $N(4) - N(8) \geq X$, since there is at least one more 4 per row than 8.
- On the other hand, counting by columns, the previous claim ensures $N(4) - N(8) \leq \frac{X}{\log_2(10)} + O_a(1)$.

And that's impossible for large X.