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Twitch Solves ISL

Episode 175

Problem

Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers c_1, c_2, c_3, ..., Turbo successively crawls distances c_1, c_2, c_3, ... around the circle, each time choosing to crawl either clockwise or counterclockwise.

Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers c_1, c_2, c_3, ... with $c_i < C$ for all i, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

External Link

https://aops.com/community/p27522975

Solution

The answer is $C = \frac{1}{2}$.

Turbo's strategy for $c_i < \frac{1}{2}$. For each i, before Turbo's i^{th} move, there is at least one point, say P_i not visited before. Because $c_i < \frac{1}{2}$, Turbo can also choose a direction to avoid P_i.

Counterexample sequence for $C > \frac{1}{2}$. Choose $\varepsilon > 0$ such that $C < \frac{1}{2} + \varepsilon$. Let

$$c_i = \begin{cases} \frac{1}{2} & i \text{ odd} \\ \frac{1}{2} + \varepsilon & i \text{ even.} \end{cases}$$

Then to avoid losing, no two consecutive moves can be in the same direction, so we can assume they go clockwise, counterclockwise, etc. Then at the end of the $(2k)^{\text{th}}$ move for $k \geq 1$, Turbo is at the endpoint of a visited interval of length $\frac{1}{2} + k\varepsilon$. Taking k large enough finishes the problem.