Twitch 174.1

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TWITCH SOLVES ISL

Episode 174

Problem

A smooth function $f\colon \mathbb{R} \to (0,\infty)$ is *laithy* if $14(\log f)'(0) \le 34(\log f)''(0)$. Prove that if f and g are laithy, then fg and $f+g$ are too.

Solution

It's clear for fg, so we focus on $f+g$. First, we rewrite the condition in terms of only the derivatives involved.

Claim. A function $h\colon \mathbb{R} \to (0,\infty)$ is laithy if and only if

$$34h'' \cdot h \ge 34(h')^2 + 14h' \cdot h$$

holds at 0.

Proof. Note that $(\log h)' = \frac{h'}{h}$ and $(\log h)'' = \frac{h''\cdot h - (h')^2}{h^2}$; hence the condition rewrites

$$14 \cdot \frac{h'}{h} \le 34 \cdot \frac{h'' \cdot h - (h')^2}{h^2}.$$ □

In what follows we evaluate always at $x=0$. Hence, the given condition is

$$\begin{aligned} 34f'' \cdot f &\ge 34(f')^2 + 14f' \cdot f \\ 34g'' \cdot g &\ge 34(g')^2 + 14g' \cdot g \end{aligned}$$

and we need to prove

$$34(f+g)'' \cdot (f+g) \ge 34(f'+g')^2 + 14(f+g)(f'+g').$$

Adding $1+\frac{g}{f}$ times the first condition and $1+\frac{f}{g}$ times the second condition gives

$$\begin{aligned} 34(f+g)'' \cdot (f+g) &\ge \left(34(f')^2 + 14f' \cdot f + 34f'^2 \cdot \frac{g}{f} + 14f'g\right) \\ &\quad + \left(34(g')^2 + 14g' \cdot g + 34g'^2 \cdot \frac{f}{g} + 14g'f\right) \\ &= 34(f')^2 + 34(g')^2 + 34\left(f'^2 \cdot \frac{g}{f} + g'^2 \cdot \frac{f}{g}\right) + 14(f+g)(f'+g') \\ &\ge 34(f')^2 + 34(g')^2 + 34f'g' + 14(f+g)(f'+g') \\ &= 34(f'+g')^2 + 14(f+g)(f'+g') \end{aligned}$$

by AM-GM on the positive numbers $f'(0)^2$, $f(0)$, $g'(0)^2$, $g(0)$. This completes the proof.