RMM 2026/3

Evan Chen

Twitch Solves ISL

Episode 174

Problem

Let $\mathcal{S}$ be a finite subset of $\mathbb{R}^3$. Prove that there exists $P, Q, R \in \mathbb{R}[x, y, z]$, such that a triple of real numbers (a, b, c) is in $\mathcal{S}$ if and only if the system of equations

$$P(x, y, z) = a, \qquad Q(x, y, z) = b, \qquad R(x, y, z) = c$$

does not have a solution $(x, y, z) \in \mathbb{R}^3$.

External Link

https://aops.com/community/p37324679

Solution

We start by constructing the function

$$F\colon \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad (x,y,z) \mapsto \big(xy-1, -[x^4(xy-1)+x^2z^2+y], z\big).$$

Claim. The function F achieves every point except $\mathbf{0}=(0,0,0)$. In addition, for points (a,b,c) with $a,b>0$, F has exactly one pre-image for (a,b,c).

Proof. Since $z=c$, we are essentially considering the system

$$\begin{aligned} a &= xy-1 \\ -b &= ax^4+c^2x^2+y. \end{aligned}$$

If $a=b=c=0$, it's clear there can be no solution. On the other hand, the second equation implies

$$ax^5+c^2x^3+bx+(a+1)=0.$$

So, if any of a, b, c are nonzero, we get an odd-degree polynomial in x which manifestly should have at least one real solution in x. Such a value of x will then yield a suitable value of y.

In addition, the value of x is unique as long as $a>0$ and $b>0$ since in that case $a+1\neq 0$ and the quintic has a unique real solution by Descartes rule of signs. □

We proceed by induction on $\mathcal{S}$ now. For the base case $\mathcal{S}=\{\vec{v}\}$, we choose just

$$\vec{x}\mapsto F(\vec{x})+\vec{v}.$$

So assume $\mathcal{S}=\{\vec{v}_0,\vec{v}_1,\dots,\vec{v}_n\}$, with $n\geq 1$. We apply several transformations:

- WLOG (by suitable affine transformation) assume all the x, y, z coordinates of $\mathcal{S}$ are distinct.
- Change labels so that $\vec{v}_0=(x_0,y_0,z_0)$ is the vector with the lowest x-coordinate. Then by translation, assume $\vec{v}_0=\mathbf{0}$, so all other x-coordinates are positive.
- Finally, apply a shear transformation of the form

  $$(x,y,z)\mapsto(x,y+Mx,z)$$

  on all the points, for M large enough, such that all y-coordinates are positive.

The critical insight is that because of our shear transformation, for each i there is a unique pre-image $\vec{w}_i$ satisfying $F(\vec{w}_i)=\vec{v}_i$. Appeal to the inductive hypothesis to find a function G whose image is exactly $\mathbb{R}^3-\{\vec{w}_1,\dots,\vec{w}_n\}$. Hence, the inductive step is achieved by the function

$$\vec{x}\mapsto F(G(\vec{x})).$$