EGMO 2022/5

Evan Chen

Twitch Solves ISL

Episode 174

Problem

For all positive integers n, k, let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.) Find all positive integers n such that for every positive integer k, the number $f(n, 2k)$ is odd.

External Link

https://aops.com/community/p24921862

Solution

The answer is $n = 2^\ell - 1$, for all $\ell \geq 1$.

We prove the following three claims:

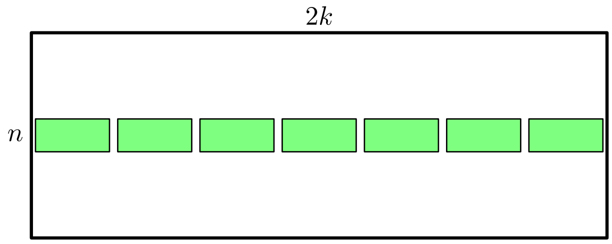
Claim. $f(1, 2k) = 1$ for all $k \geq 1$.

Proof. Immediate. □

Claim. For $k \geq 1$ and odd $n > 1$, we have

$$f(n, 2k) \equiv f\left(\frac{n-1}{2}, 2k\right) \pmod 2.$$

Proof. In any tiling of an $n \times 2k$ rectangle, we consider reflection along the side of length n (say the height). This pairs up every tiling in an involution; the fixed points are those for which the center row is a belt of horizontal dominoes.



Hence, the fixed points of the involution correspond to tiling the $\frac{n-1}{2} \times 2k$ upper half (say) and then reflecting. □

When $n = 2^\ell - 1$, this already implies that all such n have the property. Conversely, for the remaining values of n, it would suffice to show that for every *even* integer m, there exists k such that $f(m, 2k)$ is even. This is done with the following:

Claim. For each $k \geq 1$, $f(2k, 2k)$ is even.

Proof. Reflect along the main diagonal of the square to pair all valid tilings. □