Mexico 2025/5

Evan Chen

Twitch Solves ISL

Episode 173

Problem

Determine for which positive integers $n \geq 3$ there exist n not necessarily distinct prime numbers p_1, p_2, ..., p_n such that

$$p_1p_2 + 1, p_2p_3 + 1, \ldots, p_{n-1}p_n + 1 \quad \text{and} \quad p_np_1 + 1$$

are all perfect powers.

External Link

https://aops.com/community/c6h3719249

Solution

The answer is even n only. Note that a construction given by $(2, 13, 2, 13, 2, 13, 2, 13, \dots)$.

We always work with indices modulo n and prove the following. (Note that we do not need the hypothesis n is odd yet; the following claim is for any three consecutive primes.)

Claim. We have:

- If p_i and p_{i+1} are odd primes, then $p_i \not\equiv p_{i+1} \pmod 4$.
- If $p_i = 2$ then $p_{i-1} \equiv p_{i+1} \equiv 1 \pmod 4$.

Proof. If p_i and p_{i+1} are both odd, then $p_i p_{i+1} + 1$ is an even perfect power and hence divisible by 4.

Now suppose $p_i = 2$. Then

$$2p_{i-1} = x^r - 1 = (x-1)(x^{r-1} + x^{r-2} + \cdots + 1)$$

for some $r, x \ge 2$. Since the second factor is obviously greater than 2, we must have $x = 3$, and also r must be an prime. Note that $r \ne 2$ as well, since $3^2 - 1 = 8$. Hence, r is in fact odd and $3^r - 1 \equiv 2 \pmod 8$, so $p_{i-1} \equiv 1 \pmod 4$. The proof that $p_{i+1} \equiv 1 \pmod 4$ is analogous. □

Since 2's may never be adjacent, when n is odd we may assume WLOG that p_1 and p_2 are both odd, say with $p_1 \equiv 3 \pmod 4$ and $p_2 \equiv 1 \pmod 4$. Then iterating the claim above on (p_2, p_3), (p_3, p_4) and so on we get the following inductive claim:

Claim. For every $i \ge 1$, either $p_i = 2$ or $p_i \equiv (-1)^i \pmod 4$.

With indices modulo n, this forces n to be even.