EGMO 2022/4

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Twitch Solves ISL

Episode 172

Problem

Given a positive integer $n \geq 2$, determine the largest positive integer N for which there exist $N + 1$ real numbers $a_0, a_1, \ldots, a_N$ such that

(1) $a_0 + a_1 = -\frac{1}{n}$, and

(2) $(a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ for $1 \leq k \leq N - 1$.

External Link

https://aops.com/community/p24921874

Solution

The answer is $N = n$.

Let $b_i := a_i + a_{i-1}$. Then the given condition is

$$b_k b_{k+1} = b_k - b_{k+1} \implies b_{k+1} = \frac{b_k}{b_k + 1}.$$

So we can recursively compute every b_i:

$$\begin{aligned} b_2 &= -\frac{1}{n-1} \\ b_3 &= -\frac{1}{n-2} \\ &\vdots \\ b_n &= -1. \end{aligned}$$

We thus cannot have an a_{n+1} in the sequence, because we'd get a contradiction. Hence $N \le n$, and a construction by setting $a_0 = 0$ and recursively setting $a_i = -\frac{1}{n+1-i} - a_{i-1}$.