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Episode 172

Problem

Let $P_1 < P_2 < P_3 < \dots$ be the ordered sequence consisting of all positive integers that are products of two (not necessarily distinct) Fibonacci numbers. The first few terms are $1, 2, 3, 4, 5, 6, 8, 9, 10, 13, \dots$. Is $P_{i+1} - P_i$ always a Fibonacci number?

External Link

https://aops.com/community/p13026347

Solution

The answer is yes, it's always a Fibonacci number.

Index the Fibonacci numbers by $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, and so on. We say a product of the form F_iF_j is in *tier* $i + j$. The basic claim is that all the tiers are disjoint from each other except at the endpoints. To give a concrete demonstration, we list the numbers in tiers 7, 8, 9, 10, 11, 12, 13 in the following table:

$$\begin{aligned}
&\underbrace{F_2F_5}_{1\cdot 5=5} < \underbrace{F_3F_4}_{2\cdot 3=6} < \underbrace{F_1F_6}_{1\cdot 8=8} \\
&= \underbrace{F_2F_6}_{1\cdot 8=8} < \underbrace{F_4F_4}_{3\cdot 3=9} < \underbrace{F_3F_5}_{2\cdot 5=10} < \underbrace{F_1F_7}_{1\cdot 13=13} \\
&= \underbrace{F_2F_7}_{1\cdot 13=13} < \underbrace{F_4F_5}_{3\cdot 5=15} < \underbrace{F_3F_6}_{2\cdot 8=16} < \underbrace{F_1F_8}_{1\cdot 21=21} \\
&= \underbrace{F_2F_8}_{1\cdot 21=21} < \underbrace{F_4F_6}_{3\cdot 8=24} < \underbrace{F_5F_5}_{5\cdot 5=25} < \underbrace{F_3F_7}_{2\cdot 13=26} < \underbrace{F_1F_9}_{1\cdot 34=34} \\
&= \underbrace{F_2F_9}_{1\cdot 34=34} < \underbrace{F_4F_7}_{3\cdot 13=39} < \underbrace{F_5F_6}_{5\cdot 8=40} < \underbrace{F_3F_8}_{2\cdot 21=42} < \underbrace{F_1F_{10}}_{1\cdot 55=55} \\
&= \underbrace{F_2F_{10}}_{1\cdot 55=55} < \underbrace{F_4F_8}_{3\cdot 21=63} < \underbrace{F_6F_6}_{8\cdot 8=64} < \underbrace{F_5F_7}_{5\cdot 13=65} < \underbrace{F_3F_9}_{2\cdot 34=68} < \underbrace{F_1F_{11}}_{1\cdot 89=89} \\
&= \underbrace{F_2F_{11}}_{1\cdot 89=89} < \underbrace{F_4F_9}_{3\cdot 34=102} < \underbrace{F_6F_7}_{8\cdot 13=104} < \underbrace{F_5F_8}_{5\cdot 21=105} < \underbrace{F_3F_{10}}_{2\cdot 55=110} < \underbrace{F_1F_{12}}_{1\cdot 144=144} .
\end{aligned}$$

We have the following general identities:

Lemma.

- For $i \ge 1$ we have $F_{i-1}F_{i+1} - F_i = (-1)^i$.
- For $0 \le i \le j$ we have $F_{i+2}F_j - F_iF_{j+2} = (-1)^iF_{j-i}$.

Proof. Annoying calculation with Binet formula. □

More generally, for this problem we only need two claims, both of which can be proven directly.

Claim. Let n be an even integer. Then the complete list of products at tier n is

$$F_2F_{n-2} < F_4F_{n-4} < \cdots < F_{n/2}F_{n/2} < F_5F_{n-5} < F_3F_{n-3} < F_1F_{n-1}$$

and the consecutive differences are positive Fibonacci numbers.

Similarly, if n is an even integer, we have the same statement for

$$F_2F_{n-2} < F_4F_{n-4} < \cdots < F_{n/2}F_{n/2} < F_5F_{n-5} < F_3F_{n-3} < F_1F_{n-1}$$

It remains only to note that the tiers overlap exactly at endpoints, and this completes the proof.

Remark. Quote from audience member: "if this was on a [competitive programming] contest it would ask for the nth number in this sequence and then Evan would WA like 4 times due to various off by one errors".