Serbia TST 2024/6

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Twitch Solves ISL

Episode 167

Problem

In the plane, there is a figure in the form of an L-tromino, which is composed of 3 unit squares, which we will denote by Φ_0. On every move, we choose an arbitrary straight line in the plane and using it we construct a new figure. The figure Φ_n, obtained in the n^{th} move, is obtained as the union of the figure Φ_{n-1} and its axial reflection with respect to the chosen line. For the move to be valid, it is necessary that the area of the newly obtained piece is twice as large as the previous one. Is it possible to cover the whole plane in that process?

External Link

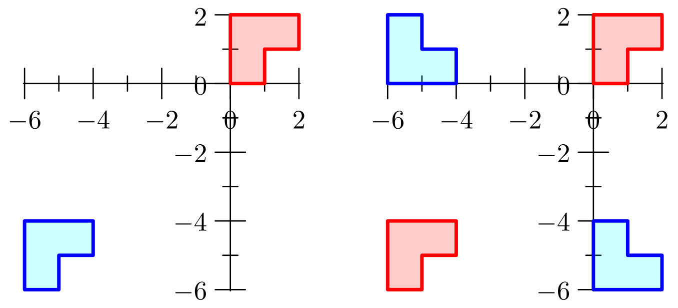
https://aops.com/community/p30734471

Solution

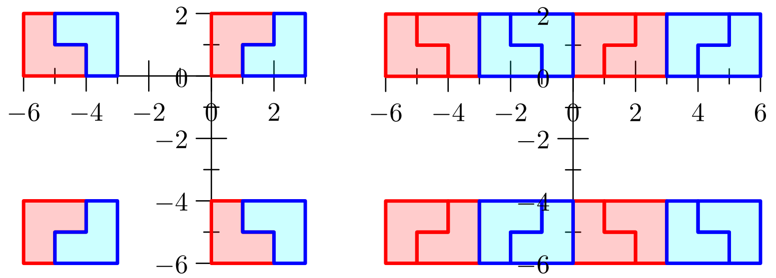
Yes, it's possible!

Situation Φ_0 with coordinates $(0,0)$, $(0,2)$, $(2,2)$, $(2,1)$, $(1,1)$, $(1,0)$ as shown below. Start with the following series of moves:

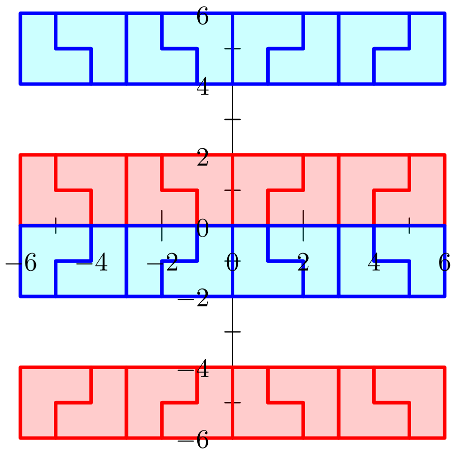
1. Start by reflecting around $x+y=-4$, then $y=-2$:



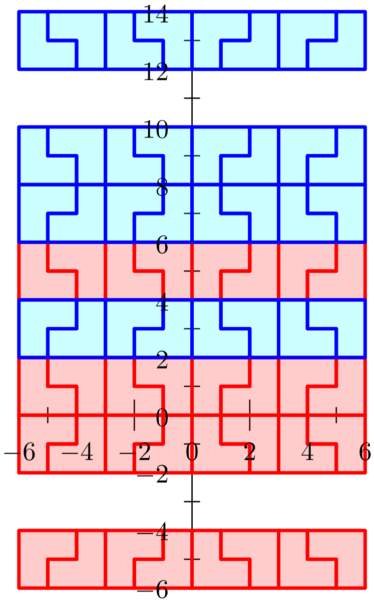
2. Reflect around $x=-\frac{3}{2}$, then $x=0$:



3. Then reflection around $y=0$:



4. Then reflection around $y=4$:



In general, we may then construct a sequence of steps which fills in the gap of length 2 (so the next one would be at $y = 12$, etc.). Meanwhile, we may also reflect around suitable values of x to extend in the y-direction. In this way complete a tessellation of the entire plane.