EGMO 2022/2

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Twitch Solves ISL

Episode 167

Problem

Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, the following two conditions hold:

(1) $f(ab) = f(a)f(b)$, and

(2) at least two of the numbers $f(a)$, $f(b)$, and $f(a+b)$ are equal.

External Link

https://aops.com/community/p24921879

Solution

The answer is

$$f(n) = c^{\nu_p(n)}$$

for any positive integer c and any fixed prime p. It's easy to check this works, because when $\nu_p(a) \neq \nu_p(b)$ we get $\nu_p(a+b) = \min\{\nu_p(a), \nu_p(b)\}$.

We now check these are the only solutions. Because f is completely multiplicative, if there is at most one prime p with $f(p) > 1$, then f i s already of the form described. So suppose now (for contradiction) $p < q$ are the two smallest primes for which $f(p) > 1$ and $f(q) > 1$.

First, invoke the problem statement on

$$f(q-p) = 1, \qquad f(p), \qquad f(q).$$

where $f(q-p) = 1$ because $q - p$ is not divisible by p. We thus conclude $f(p) = f(q)$.

Now we would get the desired contradiction upon proving

Claim. We can pick positive integers a and b such that

$$\begin{aligned} \nu_p(a+b) &\geq 2 \\ \nu_p(a) = \nu_p(b) &= 0 \\ \nu_q(b) &= 1 \\ \nu_p(a) = \nu_p(a+b) &= 0. \end{aligned}$$

Proof. Let $n := \lceil \log_p q \rceil \geq 2$; and commit to choosing

$$a + b = p^n.$$

Then we pick b to be the largest multiple of q we can: that is, if we define

$$\ell := \left\lfloor \frac{p^n}{q} \right\rfloor < \frac{p^n}{q} < q$$

then we pick

$$b = q \cdot \ell$$

so that $a = p^n - b < q$ is the remainder when p^n is divided by q. Since $\ell < q$, we have $\nu_q(b) = 1 + \nu_q(\ell) = 1$ and it's now clear that all the desired properties hold. $\square$